

POINCARÉ MEETS KORN VIA MAXWELL: EXTENDING KORN'S FIRST INEQUALITY TO INCOMPATIBLE TENSOR FIELDS

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Dedicated to Rolf Leis on the occasion of his 80th birthday

Abstract

For a bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary Γ and some relatively open Lipschitz subset $\Gamma_t \neq \emptyset$ of Γ , we prove the existence of some $c > 0$, such that

$$c \|T\|_{L^2(\Omega, \mathbb{R}^{3 \times 3})} \leq \|\operatorname{sym} T\|_{L^2(\Omega, \mathbb{R}^{3 \times 3})} + \|\operatorname{Curl} T\|_{L^2(\Omega, \mathbb{R}^{3 \times 3})} \quad (0.1)$$

holds for all tensor fields in $H(\operatorname{Curl}; \Omega)$, i.e., for all square-integrable tensor fields $T : \Omega \rightarrow \mathbb{R}^{3 \times 3}$ with square-integrable generalized rotation $\operatorname{Curl} T : \Omega \rightarrow \mathbb{R}^{3 \times 3}$, having vanishing restricted tangential trace on Γ_t . If $\Gamma_t = \emptyset$, (0.1) still holds at least for simply connected Ω and for all tensor fields $T \in H(\operatorname{Curl}; \Omega)$ which are $L^2(\Omega)$ -perpendicular to $\mathfrak{so}(3)$, i.e., to all skew-symmetric constant tensors. Here, both operations, Curl and tangential trace, are to be understood row-wise.

For compatible tensor fields $T = \nabla v$, (0.1) reduces to a non-standard variant of the well known Korn's first inequality in \mathbb{R}^3 , namely

$$c \|\nabla v\|_{L^2(\Omega, \mathbb{R}^{3 \times 3})} \leq \|\operatorname{sym} \nabla v\|_{L^2(\Omega, \mathbb{R}^{3 \times 3})}$$

for all vector fields $v \in H^1(\Omega, \mathbb{R}^3)$, for which ∇v_n , $n = 1, \dots, 3$, are normal at Γ_t . On the other hand, identifying vector fields $v \in H^1(\Omega, \mathbb{R}^3)$ (having the proper boundary conditions) with skew-symmetric tensor fields T , (0.1) turns to Poincaré's inequality since

$$\sqrt{2}c \|v\|_{L^2(\Omega, \mathbb{R}^3)} = c \|T\|_{L^2(\Omega, \mathbb{R}^{3 \times 3})} \leq \|\operatorname{Curl} T\|_{L^2(\Omega, \mathbb{R}^{3 \times 3})} \leq 2 \|\nabla v\|_{L^2(\Omega, \mathbb{R}^3)}.$$

Therefore, (0.1) may be viewed as a natural common generalization of Korn's first and Poincaré's inequality. From another point of view, (0.1) states that one can omit compatibility of the tensor field T at the expense of measuring the deviation from compatibility through $\operatorname{Curl} T$. Decisive tools for this unexpected estimate are

the classical Korn's first inequality, Helmholtz decompositions for mixed boundary conditions and the Maxwell estimate.

Key Words Korn's inequality, incompatible tensors, Maxwell's equations, Helmholtz decomposition, Poincaré type inequalities, Friedrichs-Gaffney inequality, mixed boundary conditions, tangential traces

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1 Introduction

In this contribution we show that Korn's first inequality can be generalized in some not so obvious directions, namely to tensor fields which are not gradients. Our study is a continuation from [82, 81, 84, 83] and here we generalize our results to weaker boundary conditions and domains of more complicated topology. For the proof of our main inequality (0.1) we combine techniques from electro-magnetic and elastic theory, namely

(HD) Helmholtz' decomposition,

(MI) the Maxwell inequality,

(KI) Korn's inequality.

Since these three tools are crucial for our results we briefly look at their history. As pointed out in the overview [110], Helmholtz founded a comprehensive development in the theory of projections methods mostly applied in, e.g., electromagnetic or elastic theory or fluid dynamics. His famous theorem HD, see Lemma 3, states, that any sufficiently smooth and sufficiently fast decaying vector field can be characterized by its rotation and divergence or can be decomposed into an irrotational and a solenoidal part. A first uniqueness result was given by Blumenthal in [8]. Later, Hilbert and Banach space methods have been used to prove similar and refined decompositions of the same type.

The use of inequalities is widespread in establishing existence and uniqueness of solutions of partial differential equations. Furthermore, often these inequalities ensure that the solution is in a more suitable space from a numerical view point than the solution space itself. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz continuous boundary Γ . Moreover, let Γ_t, Γ_n be some relatively open Lipschitz subsets of Γ with $\overline{\Gamma_t} \cup \overline{\Gamma_n} = \Gamma$ and $\Gamma_t \neq \emptyset$. In potential theory use is made of **Poincaré's inequality**, this is

$$\|u\|_{L^2(\Omega)} \leq c_p \|\nabla u\|_{L^2(\Omega)} \quad (1.1)$$

for all functions $u \in \mathring{H}^1(\Gamma_t; \Omega)^1$ with some constant $c_p > 0^2$, to bound the scalar potential in terms of its gradient. In elasticity theory **Korn's first inequality** in combination with Poincaré's inequality, this is

$$(c_p^2 + 1)^{-1/2} \|v\|_{H^1(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)} \leq c_k \|\text{sym } \nabla v\|_{L^2(\Omega)} \quad (1.2)$$

for all vector fields $v \in \mathring{H}^1(\Gamma_t; \Omega)$ with some constant $c_k > 0$, is needed for bounding the deformation of an elastic medium in terms of the symmetric strains, i.e., the symmetric part $\text{sym } \nabla v = \frac{1}{2}(\nabla v + (\nabla v)^\top)$ of the Jacobian ∇v . In electro-magnetic theory the **Maxwell inequality** (see Lemma 1), this is

$$\|v\|_{L^2(\Omega)} \leq c_m (\|\text{curl } v\|_{L^2(\Omega)} + \|\text{div } v\|_{L^2(\Omega)}) \quad (1.3)$$

for all $v \in \mathring{H}(\text{curl}; \Gamma_t, \Omega) \cap \mathring{H}(\text{div}; \Gamma_n, \Omega) \cap \mathcal{H}(\Omega)^\perp$ with some positive c_m , is used to bound the electric and magnetic field in terms of the electric charge and current density, respectively. Actually, this important inequality is just the continuity estimate of the corresponding electro-magnto static solution operator. It has different names in the literature, e.g., Friedrichs', Gaffney's or Poincaré type inequality [38, 32].

It is well known that Korn's and Poincaré's inequality are not equivalent. However, one main result of our paper is that both inequalities, i.e., (1.1), (1.2), can be inferred from the more general result (0.1), where (1.3) is used within the proof.

¹For exact definitions see section 2.

²In the following $c_p, c_k, c_m > 0$ refer to the constants in Poincaré's, Korn's and in the Maxwell inequalities, respectively.

1.1 The Maxwell inequality

Concerning the MI (Lemma 1) in 1968 Leis [66] considered the boundary value problem of total reflection for the inhomogeneous and anisotropic Maxwell system as well in bounded as in exterior domains. For bounded domains $\Omega \subset \mathbb{R}^3$ he was able to estimate the derivatives of vector fields v by the fields themselves, their divergence and their rotation in $L^2(\Omega)$, i.e.,

$$c \sum_{n=1}^3 \|\partial_n v\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)} + \|\operatorname{curl} v\|_{L^2(\Omega)} + \|\operatorname{div} v\|_{L^2(\Omega)}, \quad (1.4)$$

provided that the boundary Γ is sufficiently smooth and that $\nu \times v|_{\Gamma} = 0^3$, i.e., the tangential trace of v vanishes at Γ . Of course, (1.4) implies

$$c \|v\|_{H^1(\Omega)} \leq \|v\|_{L^2(\Omega)} + \|\operatorname{curl} v\|_{L^2(\Omega)} + \|\operatorname{div} v\|_{L^2(\Omega)}$$

and thus by Rellich's selection theorem the **Maxwell compactness property** (MCP), i.e.,

$$\begin{aligned} X(\Omega) &:= \overset{\circ}{H}(\operatorname{curl}; \Omega) \cap H(\operatorname{div}; \Omega) \\ &= \{v \in L^2(\Omega) : \operatorname{curl} v \in L^2(\Omega), \operatorname{div} v \in L^2(\Omega), \nu \times v|_{\Gamma} = 0\} \end{aligned}$$

is compactly embedded into $L^2(\Omega)$, since $X(\Omega)$ is a closed subspace of the Sobolev-Hilbert space $H^1(\Omega)$. However, (1.4), which is often called Friedrichs' or Gaffney's inequality, fails if smoothness of $\partial\Omega$ is not assumed. On the other hand, by a standard indirect argument the MCP implies the Maxwell inequality (1.3) for $\Gamma_t = \Gamma$. Hence, the compact embedding

$$X(\Omega) \hookrightarrow L^2(\Omega) \quad (1.5)$$

is crucial for a solution theory suited for Maxwell's equations as well as for the validity of the Maxwell estimate (1.3) or Lemma 1. But in the non-smooth case compactness of (1.5) can not be proved by Rellich's selection theorem. On the other hand, if (1.5) is compact, one obtains Fredholm's alternative for time-harmonic/static Maxwell equations and the Maxwell inequality for bounded domains. For unbounded domains, e.g., exterior domains, (local) compactness implies Eidus' limiting absorption and limiting amplitude principles and the corresponding weighted Maxwell inequalities [26, 27, 28, 25]. These are the right and crucial tools for treating radiation problems, see the papers by Pauly [89, 90, 92, 91, 64] for the latest results. Therefore, Leis encouraged some of his students to deal with electro-magnetic problems, in particular with the MCP-question, see [95, 96, 97, 98, 99, 100, 108, 118, 122].

In 1969 Rinkens [108] (see also [67]) presented an example of a non-smooth domain where the embedding of $X(\Omega)$ into $H^1(\Omega)$ is not possible. Another example had been found shortly later and is written down in a paper by Saranen [109].

³ ν denotes the outward unit normal at Γ and \times respectively \cdot the vector respectively scalar product in \mathbb{R}^3 .

Henceforth, there was a search for proofs which do not make use of an embedding of $X(\Omega)$ into $H^1(\Omega)$. In 1974 Weck [118] obtained a first and quite general result for ‘cone-like’ regions. Weck considered a generalization of Maxwell’s boundary value problem to Riemannian manifolds of arbitrary dimension N , going back to Weyl [121]. The cone-like regions have Lipschitz boundaries but maybe not the other way round. However, polygonal boundaries are covered by Weck’s result. In a joint paper by Picard, Weck and Witsch [100] Weck’s proof has been modified to obtain (1.5) even for domains which fail to have Lipschitz boundary.

Other proofs of (1.5) for Lipschitz domains have been given by Costabel [20] and Weber [116]. Costabel showed that $X(\Omega)$ is already embedded into the fractional Sobolev space $H^{1/2}(\Omega)$. Weber’s proof has been modified by Witsch [122] to obtain the result for domains with Hölder continuous boundaries (with exponent $p > 1/2$). Finally, there is a quite elegant result by Picard [97] who showed that if the result holds for smooth boundaries it holds for Lipschitz boundaries as well. This result remains true even in the generalized case (for Riemannian manifolds).

In this paper we shall make use of a result by Jochmann [52] who allows a Lipschitz boundary Γ which is divided into two parts Γ_t and Γ_n by a Lipschitz curve and such that on Γ_t and Γ_n the mixed boundary conditions $\nu \times v|_{\Gamma_t} = 0$ and $\nu \cdot v|_{\Gamma_n} = 0$ respectively, hold. In his dissertation, Kuhn [63] has proved an analogous result for the generalized Maxwell equations on Riemannian manifolds, following Weck’s approach.

The well known Sobolev type space

$$H(\text{curl}; \Omega)$$

has plenty of important and prominent applications, most of them in the comprehensive theory of Maxwell’s equations, i.e., in electro-magnetic theory. Among others, we want to mention [66, 65, 67, 95, 96, 97, 98, 99, 118, 120, 122, 116, 117, 89, 90, 92, 91, 64, 93]. It is also used as a main tool for the analysis and discretization of Navier-Stokes’ equations and in the numerical analysis of non-conforming finite element discretizations [43, 41].

1.2 Korn’s inequality

Korn’s inequality gives the control of the $L^2(\Omega)$ -norm of the gradient of a vector field by the $L^2(\Omega)$ -norm of just the symmetric part of its gradient, under certain conditions. The most elementary variant of Korn’s inequality for $\Gamma_t = \Gamma$ reads as follows: For any smooth vector field $v : \Omega \rightarrow \mathbb{R}^3$ with compact support in Ω , i.e., $v \in \mathring{C}^\infty(\Omega)$,

$$\|\nabla v\|_{L^2(\Omega)}^2 \leq 2 \|\text{sym } \nabla v\|_{L^2(\Omega)}^2 \quad (1.6)$$

holds. This inequality is simply obtained by straight forward partial integration, see the Appendix, and dates back to Korn himself [61]. Moreover, it can be improved easily by estimating just the deviatoric part of the symmetric gradient (see Appendix), this is

$$\forall v \in \mathring{H}^1(\Omega) \quad \frac{1}{2} \|\nabla v\|_{L^2(\Omega)}^2 \leq \|\text{dev sym } \nabla v\|_{L^2(\Omega)}^2 \leq \|\text{sym } \nabla v\|_{L^2(\Omega)}^2. \quad (1.7)$$

Here, we introduce generally the deviatoric part

$$\operatorname{dev} T := \operatorname{dev}_{1/3} T := T - \frac{1}{3} \operatorname{tr} T \operatorname{id}$$

and the symmetric and skew-symmetric parts

$$\operatorname{sym} T := \frac{1}{2}(T + T^\top), \quad \operatorname{skew} T := \frac{1}{2}(T - T^\top)$$

for quadratic matrix or tensor fields T . Note that $T = \operatorname{sym} T + \operatorname{skew} T$ and $\operatorname{sym} T$, $\operatorname{skew} T$ are orthogonal in $\mathbb{R}^{3 \times 3}$. Together with (component-wise) Poincaré's inequality (1.1) for $\Gamma_t = \Gamma$, see [102], one arrives as in (1.2) for all $v \in \mathring{\mathbf{H}}^1(\Omega)$ at

$$(c_p^2 + 1)^{-1/2} \|v\|_{\mathbf{H}^1(\Omega)} \leq \|\nabla v\|_{\mathbf{L}^2(\Omega)} \leq \sqrt{2} \|\operatorname{dev} \operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} \leq \sqrt{2} \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}.$$

Then, Rellich's selection theorem shows that the set of all $\mathring{\mathbf{H}}^1(\Omega)$ -vector fields whose (deviatoric) symmetric gradients are bounded in $\mathbf{L}^2(\Omega)$ is (sequentially) compact in $\mathbf{L}^2(\Omega)$.

Let us mention that Arthur Korn (1870-1945) was a student of Poincaré. Korn visited him in Paris before the turn of the 20th century and it was again Korn who wrote the obituary for Poincaré in 1912 [62]. It is also worth mentioning that Poincaré helped to introduce Maxwell's electro-magnetic theory to french readers. The interesting life of the german-jewish mathematician, physicist and inventor of telegraphy Korn is recalled in [70, 101].

In general, Korn's inequality involves an integral measure of shape deformation, i.e., a measure of strain $\|\operatorname{sym} \nabla v\|$, with which it is possible to control the distance of the deformation to some Euclidean motion or to control the $\mathbf{H}^1(\Omega)$ -norm or semi-norm.

Consider the kernel of the linear operator $\operatorname{sym} \nabla : \mathbf{H}^1(\Omega) \subset \mathbf{L}^2(\Omega) \rightarrow \mathbf{L}^2(\Omega)$

$$\ker(\operatorname{sym} \nabla) = \mathbf{RM} := \{x \mapsto Ax + b : A \in \mathfrak{so}(3), b \in \mathbb{R}^3\}, \quad (1.8)$$

the space of all infinitesimal rigid displacements (motions) which consists of all affine linear transformations v for which $\nabla v = A \in \mathfrak{so}(3)$, i.e., A is skew-symmetric⁴. Since the measure of strain $\operatorname{sym} \nabla v$ is invariant with respect to superposed infinitesimal rigid displacements, i.e., $\mathbf{RM} \subset \ker(\operatorname{sym} \nabla)$, one needs some linear boundary or normalization conditions in order to fix this Euclidean motion. E.g., using homogeneous Dirichlet boundary conditions one has (1.2). By normalization one gets

$$\|\nabla v\|_{\mathbf{L}^2(\Omega)} \leq c_k \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} \quad (1.9)$$

⁴(1.8) easily follows from the simple observation that $\operatorname{sym} \nabla v = 0$ implies $\nabla v(x) = A(x) \in \mathfrak{so}(3)$. Taking the Curl on both sides gives $\operatorname{Curl} A = 0$ and thus $\nabla A = 0$. Hence, A must be a constant skew-symmetric matrix. Equivalently, one may use the well known representation for second derivatives $\partial_i \partial_j v_k = \partial_j (\operatorname{sym} \nabla v)_{ik} + \partial_i (\operatorname{sym} \nabla v)_{jk} - \partial_k (\operatorname{sym} \nabla v)_{ij}$. Then, $\operatorname{sym} \nabla v = 0$ implies that v is a first order polynomial. This representation formula for second derivatives of v in terms of derivatives of strain components can also serve as basis for a proof of Korn's second inequality [17, 24]. In this case one uses the lemma of Lions, see [16], i.e., for a Lipschitz domain $u \in \mathbf{L}^2(\Omega)$ if and only if $u \in \mathbf{H}^{-1}(\Omega)$ and $\nabla u \in \mathbf{H}^{-1}(\Omega)$.

for all $v \in \mathbf{H}^1(\Omega)$ with $\nabla v \perp \mathfrak{so}(3)$ ⁵. Equivalently, one has for all $v \in \mathbf{H}^1(\Omega)$, e.g.,

$$\|\nabla v - A_{\nabla v}\|_{\mathbf{L}^2(\Omega)} \leq c_k \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}, \quad (1.10)$$

where the constant skew-symmetric tensor

$$A_{\nabla v} := \text{skew} \oint_{\Omega} \nabla v \, d\lambda \in \mathfrak{so}(3), \quad \oint_{\Omega} u \, d\lambda := \lambda(\Omega)^{-1} \int_{\Omega} u \, d\lambda \quad (\lambda: \text{Lebesgue's measure}),$$

is the $\mathbf{L}^2(\Omega)$ -orthogonal projection of ∇v onto $\mathfrak{so}(3)$. For details we refer to the appendix. Poincaré's inequalities for vector fields by normalization read

$$\|v\|_{\mathbf{L}^2(\Omega)} \leq c_p \|\nabla v\|_{\mathbf{L}^2(\Omega)}, \quad \|v\|_{\mathbf{H}^1(\Omega)} \leq (1 + c_p^2)^{1/2} \|\nabla v\|_{\mathbf{L}^2(\Omega)} \quad (1.11)$$

for all $v \in \mathbf{H}^1(\Omega)$ with $v \perp \mathbb{R}^3$. Equivalently, one has for all $v \in \mathbf{H}^1(\Omega)$, e.g.,

$$\|v - a_v\|_{\mathbf{L}^2(\Omega)} \leq c_p \|\nabla v\|_{\mathbf{L}^2(\Omega)}, \quad \|v - a_v\|_{\mathbf{H}^1(\Omega)} \leq (1 + c_p^2)^{1/2} \|\nabla v\|_{\mathbf{L}^2(\Omega)}, \quad (1.12)$$

where the constant vector

$$a_v := \oint_{\Omega} v \, d\lambda \in \mathbb{R}^3$$

is the $\mathbf{L}^2(\Omega)$ -orthogonal projection of v onto \mathbb{R}^3 . Combining (1.9) and (1.11) we obtain

$$(1 + c_p^2)^{-1/2} \|v\|_{\mathbf{H}^1(\Omega)} \leq \|\nabla v\|_{\mathbf{L}^2(\Omega)} \leq c_k \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)} \quad (1.13)$$

for all $v \in \mathbf{H}^1(\Omega)$ with $\nabla v \perp \mathfrak{so}(3)$ and $v \perp \mathbb{R}^3$. Without these conditions one has

$$(1 + c_p^2)^{-1/2} \|v - r_v\|_{\mathbf{H}^1(\Omega)} \leq \|\nabla v - A_{\nabla v}\|_{\mathbf{L}^2(\Omega)} \leq c_k \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}, \quad (1.14)$$

for all $v \in \mathbf{H}^1(\Omega)$, where the rigid motion $r_v := A_{\nabla v} \xi + a_v - A_{\nabla v} a_\xi \in \mathbf{RM}$ with the identity function $\xi(x) := \text{id}(x) = x$ reads

$$r_v(x) := A_{\nabla v} x + \oint_{\Omega} v \, d\lambda - A_{\nabla v} \oint_{\Omega} x \, d\lambda_x.$$

Note that $u := v - r_v$ belongs to $\mathbf{H}^1(\Omega)$ with $\nabla u = \nabla v - A_{\nabla v}$ and satisfies $\nabla u \perp \mathfrak{so}(3)$ and $u \perp \mathbb{R}^3$. Hence (1.13) holds for u . Moreover, we have for $v \in \mathbf{H}^1(\Omega)$

$$r_v = 0 \quad \Leftrightarrow \quad A_{\nabla v} = 0 \wedge a_v = 0 \quad \Leftrightarrow \quad \nabla v \perp \mathfrak{so}(3) \wedge v \perp \mathbb{R}^3.$$

See the appendix for details. Conditions to eliminate some or all six rigid body modes (three infinitesimal rotations and three translations) comprise (see [3])

$$\text{skew} \int_{\Omega} \nabla v \, d\lambda = 0, \quad v|_{\Gamma_t} = 0, \quad \nabla v_n \text{ normal to } \Gamma_t.$$

⁵ \perp denotes orthogonality in $\mathbf{L}^2(\Omega)$, whose elements map into \mathbb{R} , \mathbb{R}^3 or $\mathbb{R}^{3 \times 3}$, respectively.

Korn's inequality is the main tool in showing existence, uniqueness and continuous dependence upon data in linearized elasticity theory and it has therefore plenty of applications in continuum mechanics [87, 47]. One refers usually to [59, 60, 61] for first versions of Korn's inequalities. These original papers by Korn are, however, difficult to read nowadays and Friedrichs even claims that they are wrong [31]. In any case, in [61, p.710(13)] Korn states that (in modern notation)

$$\|\text{skew } \nabla v\|_{\mathbf{L}^2(\Omega)} \leq c_k \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}$$

holds for all vector fields $v : \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3$ having Hölder continuous first order derivatives and which satisfy

$$\int_{\Omega} v \, d\lambda = 0, \quad \int_{\Omega} \text{skew } \nabla v \, d\lambda = 0.$$

Note that this implies $A_{\nabla v} = 0$ and $a_v = 0$ and hence $r_v = 0$.

Let $\Gamma_t \neq \emptyset$. By the classical **Korn's first inequality** with homogeneous Dirichlet boundary condition we mean

$$\exists c_k > 0 \quad \forall v \in \mathring{\mathbf{H}}^1(\Gamma_t; \Omega) \quad \|\nabla v\|_{\mathbf{L}^2(\Omega)} \leq c_k \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}$$

or equivalently by Poincaré's inequality (1.1)

$$\exists c_k > 0 \quad \forall v \in \mathring{\mathbf{H}}^1(\Gamma_t; \Omega) \quad \|v\|_{\mathbf{H}^1(\Omega)} \leq c_k \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)},$$

see (1.2), whereas we say that the classical **Korn's second inequality** holds if

$$\exists c_k > 0 \quad \forall v \in \mathbf{H}^1(\Omega) \quad \|v\|_{\mathbf{H}^1(\Omega)} \leq c_k (\|v\|_{\mathbf{L}^2(\Omega)} + \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}).$$

Korn's first inequality can be obtained as a consequence of Korn's second inequality⁶ and the compactness of the embedding $\mathbf{H}^1(\Omega) \hookrightarrow \mathbf{L}^2(\Omega)$, i.e., Rellich's selection theorem for $\mathbf{H}^1(\Omega)$. Thus, the main task for Korn's inequalities is to show Korn's second inequality. Korn's second inequality in turn can be seen as a strengthened version of Gårding's inequality requiring methods from Fourier analysis [44, 45, 53]. Very elegant and short proofs of Korn's second inequality have been presented in [58, 113] and by Fichera [29]. Fichera's proof can be found in the appendix of Leis' book [67]. Another short proof is based on strain preserving extension operators [86].

Both inequalities admit a natural extension to the Sobolev space $W^{1,p}(\Omega)$ for Sobolev exponents $1 < p < \infty$. The first proofs have been given by Mosolov and Mjasnikov in [72, 73] and by Ting in [114]. Note that Korn's inequalities are wrong⁷ in $W^{1,1}(\Omega)$, see

⁶The ascription Korn's first or second inequality is not universal. Friedrichs [31] refers to Korn's inequality $\|\nabla v\|_{\mathbf{L}^2(\Omega)} \leq c_k \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}$ in the first case if $u|_{\Gamma} = 0$ and to the second case if $\text{skew } \int_{\Omega} \nabla v \, d\lambda$ vanishes. We follow the usage in [115, p.54].

⁷Korn's inequalities are also wrong in $W^{1,\infty}(\Omega)$. E.g., consider the unit ball in \mathbb{R}^2 and the vector field $v(x) := \ln |x|(x_2, -x_1)$.

[88]. New and simple counterexamples for $W^{1,1}(\Omega)$ have been obtained in [19]. Friedrichs furnished the first⁸ modern proof of the above inequalities [31], see also [94, 42, 31, 44, 45, 6, 86, 58, 17, 5, 46, 48, 24]. A version of Korn's inequality for sequences of gradient young measures has been obtained in [7].

Korn's inequalities are also crucial in the finite element treatment of problems in solid mechanics with non-conforming or discontinuous Galerkin methods. Piecewise Korn's inequalities subordinate to the mesh and involving jumps across element boundaries are investigated, e.g., in [10, 68]. An interesting special case of Korn's first inequality with non-standard boundary conditions and for non-axi-symmetric domains with applications in statistical mechanics has been treated in [22].

Ciarlet [16, 15, 18] has shown how to extend Korn's inequalities to curvilinear coordinates in Euclidean space which has applications in shell theory. It is possible to extend such generalizations to more general Riemannian manifolds [13]. Korn's inequalities for thin domains with uniform constants have been investigated, e.g., in [69].

Korn's inequalities appear in the treatment of the Navier-Stokes model as well, since with the fluid velocity v in the Eulerian description the rate of the deformation tensor is given by $\text{sym } \nabla v$ which controls the viscous forces generated due to shearing motion. In this case, Korn's inequality acts in a geometrically exact description of the fluid motion and not just for the approximated linearized treatment as in linearized elasticity.

1.3 Further generalizations of Korn's inequalities

1.3.1 Poincaré-Korn type estimates

As already mentioned, it is well known that there are no $W^{1,1}(\Omega)$ -versions of Korn's inequalities [88, 19]. However, it is still possible to obtain a bound of the $L^p(\Omega)$ -norm of a vector field v even for $p = 1$ in terms of controlling the strain $\text{sym } \nabla v$ in some sense⁹. More precisely, let as usual $\text{BD}(\Omega)$ denote the space of bounded deformations, i.e., the space of all vector fields $v \in L^1(\Omega)$ such that all components of the tensor (matrix) $\text{sym } \nabla v$ (defined in the distributional sense) are measures with finite total variation. Then, the total variation measure of the distribution $\text{sym } \nabla v$ for a vector field $v \in L^1(\Omega)$ is defined by

$$|\text{sym } \nabla v|(\Omega) := \sup_{\substack{\Phi \in \mathring{C}^1(\Omega) \\ |\Phi|_{L^\infty(\Omega)} \leq 1}} |\langle v, \text{Div sym } \Phi \rangle_\Omega|$$

and $|\text{sym } \nabla v|(\Omega) = \|\text{sym } \nabla v\|_{L^1(\Omega)}$ holds if $\text{sym } \nabla v \in L^1(\Omega)$. In [55, 56] the inequalities

$$\begin{aligned} \exists c_{\mathbf{k}} > 0 \quad \forall v \in \text{BD}(\Omega) \quad & \inf_{r \in \text{RM}} \|v - r\|_{L^1(\Omega)} \leq c_{\mathbf{k}} |\text{sym } \nabla v|(\Omega), \\ \exists c_{\mathbf{k}} > 0 \quad \forall v \in L^p(\Omega), \text{sym } \nabla v \in L^q(\Omega) \quad & \inf_{r \in \text{RM}} \|v - r\|_{L^p(\Omega)} \leq c_{\mathbf{k}} \|\text{sym } \nabla v\|_{L^q(\Omega)} \end{aligned}$$

⁸The case $N = 2$ has already been proved by Friedrichs [30] in 1937.

⁹And this is indeed the type of inequality à la Poincaré's estimate that Korn intended to prove [61, p.707].

with

$$q \in [1, \infty) \setminus \{3\}, \quad p = \begin{cases} \frac{3q}{3-q} & , 1 \leq q < 3 \\ \infty & , q > 3 \end{cases}$$

have been proved. In case the displacement v has vanishing trace on Γ one has a Poincaré-Korn type inequality for $v \in \text{BD}(\Omega)$ [112]

$$\|v\|_{\text{L}^{3/2}(\Omega)} \leq c_k |\text{sym } \nabla v|(\Omega).$$

The weaker inequality with $\text{L}^1(\Omega)$ -term on the right hand side is already proved in [111, Th.1]. Moreover, as shown in [112, Th.II.2.4] it is clear that $\text{BD}(\Omega)$ is compactly embedded into $\text{L}^p(\Omega)$ for any $1 \leq p < 3/2$.

Considering Korn's second inequality one obtains, again via Rellich's selection theorem, the compact embedding of

$$\text{S}(\Omega) := \{v \in \text{L}^2(\Omega) : \text{sym } \nabla v \in \text{L}^2(\Omega)\}$$

into $\text{L}^2(\Omega)$ provided that the 'regularity result' $\text{S}(\Omega) \subset \text{H}^1(\Omega)$ holds, as already mentioned. In less regular domains, e.g., domains with cusps, Korn's second inequality and the embedding $\text{S}(\Omega) \subset \text{H}^1(\Omega)$ may fail, for counterexamples see [119, 40]. Weck [119] has shown that, however, compact embedding into $\text{L}^2(\Omega)$, i.e., the **elastic compactness property** (ECP)¹⁰, this is, the embedding

$$\text{S}(\Omega) \hookrightarrow \text{L}^2(\Omega) \tag{1.15}$$

is compact, still holds true, without the intermediate $\text{H}^1(\Omega)$ -estimate. Therefore, once more by a usual indirect argument, also in irregular (bounded) domains one has always the estimate

$$\|v\|_{\text{L}^2(\Omega)} \leq c_k \|\text{sym } \nabla v\|_{\text{L}^2(\Omega)}$$

for all $v \in \text{S}(\Omega) \cap \text{S}_0(\Omega)^\perp$, where $\text{S}_0(\Omega) := \{v \in \text{S}(\Omega) : \text{sym } \nabla v = 0\}$. Note that $\text{S}_0(\Omega)$ is finite dimensional¹¹ due to the compact embedding (1.15). Moreover, we have $\text{RM} \subset \text{S}_0(\Omega)$ but equality is not clear.

Extensions of Korn's inequalities to non-smooth domains and weighted versions for unbounded domains can be found in [57, 74, 49, 23, 1, 2]. Korn's inequalities in Orlicz spaces are treated, e.g., in [35, 9]. A reference for Korn's inequality for perforated domains and homogenization theory is [11].

1.3.2 Generalization to weaker strain measures

Also the second Korn's inequality can be generalized by using the trace free infinitesimal deviatoric strain measure. It holds

$$\|v\|_{\text{H}^1(\Omega)} \leq c_k (\|v\|_{\text{L}^2(\Omega)} + \|\text{dev sym } \nabla v\|_{\text{L}^2(\Omega)})$$

¹⁰Here, we have the same situation as in the Maxwell case, see the MCP and the MI.

¹¹Compare with $\mathcal{H}(\Omega)$ in (2.2).

for all $v \in H^1(\Omega)$. For proofs see [21, 51, 106, 107] and [35, 34, 36]. This version has found applications for Cosserat models and perfect plasticity [37].

Another generalization concerns the situation, where a dislocation based motivated generalized strain $\text{sym}(\nabla v F_p^{-1})$ is controlled. Such cases arise naturally when considering finite elasto-plasticity based on the multiplicative decomposition $F = F_e F_p$ of the deformation gradient into elastic and plastic parts [77, 76] or in elasticity problems with structural changes [54, 78] and shell models [79]. In case of plasticity, $F_p : \Omega \rightarrow \mathbb{R}^{3 \times 3}$ is the plastic deformation related to pure dislocation motion. The first result under the assumptions that $\det F_p \geq \mu > 0$ and F_p is sufficiently smooth, i.e., $F_p, F_p^{-1}, \text{Curl } F_p \in C^1(\overline{\Omega})$, has been given by Neff in [75]. In fact

$$\|v\|_{H^1(\Omega)} \leq c_k \|\text{sym}(\nabla v F_p^{-1})\|_{L^2(\Omega)} \quad (1.16)$$

holds for all $v \in \mathring{H}^1(\Gamma_t; \Omega)$ with c_k depending on F_p . This inequality has been generalized to mere continuity and invertibility of F_p in [103], while it is also known that some sort of smoothness of F_p beyond $L^\infty(\Omega)$ -control is necessary, see [103, 104, 85].

1.3.3 Korn's inequality and rigidity estimates

Recently, there has been a revived interest in so called rigidity results, which have a close connection to Korn's inequalities. With the point-wise representation

$$\begin{aligned} \text{dist}^2(\nabla v(x), \mathfrak{so}(3)) &= \inf_{A \in \mathfrak{so}(3)} |\nabla v(x) - A|^2 \\ &= \inf_{A \in \mathfrak{so}(3)} (|\text{sym } \nabla v(x)|^2 + |\text{skew } \nabla v(x) - A|^2) = |\text{sym } \nabla v(x)|^2, \end{aligned}$$

the infinitesimal rigidity result can be expressed as follows

$$\text{dist}(\nabla v, \mathfrak{so}(3)) = 0 \quad \Rightarrow \quad v \in \text{RM}. \quad (1.17)$$

Korn's first inequality can be seen as a qualitative extension of the infinitesimal rigidity result, this is, for $1 < p < \infty$ there exist constants $c_k > 0$ such that

$$\min_{A \in \mathfrak{so}(3)} \|\nabla v - A\|_{L^p(\Omega)} \leq c_k \|\text{sym } \nabla v\|_{L^p(\Omega)} = c_k \left(\int_{\Omega} \text{dist}^p(\nabla v, \mathfrak{so}(3)) d\lambda \right)^{1/p}$$

holds for all $v \in W^{1,p}(\Omega)$, see, e.g., [115]. As already seen in (1.10), in the Hilbert space case $p = 2$ the latter inequality can be made explicit with $A = A_{\nabla v}$ and in this form with $A_{\nabla v} = 0$ it is given by Friedrichs [31, p.446] and denoted as Korn's inequality in the second case.

The nonlinear version of (1.17) is the classical Liouville rigidity result, see [14, 105, 107]. It states that if an elastic body is deformed in such a way that its deformation gradient is point-wise a rotation, then the body is indeed subject to a rigid motion. In

mathematical terms we have for smooth maps φ , that if $\nabla\varphi \in \text{SO}(3)^{12}$ almost everywhere then $\nabla\varphi$ is constant, i.e.,

$$\text{dist}(\nabla\varphi, \text{SO}(3)) = 0 \quad \Rightarrow \quad \varphi(x) = Rx + b, \quad R \in \text{SO}(3), b \in \mathbb{R}^3. \quad (1.18)$$

The optimal quantitative version of Liouville's rigidity result has been derived by Friesecke, James and Müller in [33]. We have

$$\min_{R \in \text{SO}(3)} \left(\int_{\Omega} \text{dist}^p(\nabla\varphi, R) d\lambda \right)^{1/p} \leq c_{\mathbf{k}} \left(\int_{\Omega} \text{dist}^p(\nabla\varphi, \text{SO}(3)) d\lambda \right)^{1/p}. \quad (1.19)$$

As a consequence, if the deformation gradient is close to rotations, then it is in fact close to a unique rotation. A generalization to fracturing materials is stated in [12]. It is possible to infer a nonlinear Korn's inequality from (1.19), i.e.,

$$\|\nabla\varphi - \text{id}\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{k}} \left\| (\nabla\varphi)^{\top} \nabla\varphi - \text{id} \right\|_{\mathbf{L}^2(\Omega)}$$

for all $\varphi \in W^{1,4}(\Omega)$ with $\varphi = \text{id}$ on Γ and $\det \nabla\varphi > 0$, see [71] for more general statements. Another quantitative generalization of Liouville's rigidity result is the following: For all differentiable orthogonal tensor fields $R : \Omega \rightarrow \text{SO}(3)$

$$|\nabla R| \leq c |\text{Curl } R| \quad (1.20)$$

holds point-wise [80]. From (1.20) we may also recover (1.18) by assuming $R = \nabla\varphi$. It extends the simple inequality for differentiable skew-symmetric tensor fields $A : \Omega \rightarrow \mathfrak{so}(3)$

$$|\nabla A| \leq c |\text{Curl } A| \quad (1.21)$$

to $\text{SO}(3)$, i.e., to finite rotations [80].

After this introductory remarks we turn to the main part of our contribution.

2 Definitions and preliminaries

Let Ω be a bounded domain in \mathbb{R}^3 with Lipschitz boundary $\Gamma := \partial\Omega$. Moreover, let Γ_t be a relatively open subset of Γ separated from $\Gamma_n := \partial\Omega \setminus \overline{\Gamma_t}$ by a Lipschitz curve. For details and exact definitions see [52].

¹² $\text{SO}(3)$ denotes the space of orthogonal matrices with determinant 1.

2.1 Functions and vector fields

The usual Lebesgue spaces of square integrable functions, vector or tensor fields on Ω with values in \mathbb{R} , \mathbb{R}^3 or $\mathbb{R}^{3 \times 3}$, respectively, will be denoted by $L^2(\Omega)$. Moreover, we introduce the standard Sobolev spaces

$$\begin{aligned} H(\text{grad}; \Omega) &= \{u \in L^2(\Omega) : \text{grad } u \in L^2(\Omega)\}, & \text{grad} &= \nabla, \\ H(\text{curl}; \Omega) &= \{v \in L^2(\Omega) : \text{curl } v \in L^2(\Omega)\}, & \text{curl} &= \nabla \times, \\ H(\text{div}; \Omega) &= \{v \in L^2(\Omega) : \text{div } v \in L^2(\Omega)\}, & \text{div} &= \nabla \cdot. \end{aligned}$$

of functions u or vector fields v , respectively. $H(\text{grad}; \Omega)$ is usually denoted by $H^1(\Omega)$. Furthermore, we introduce their closed subspaces

$$\mathring{H}(\text{grad}; \Gamma_t, \Omega) = \mathring{H}^1(\Gamma_t; \Omega), \quad \mathring{H}(\text{curl}; \Gamma_t, \Omega), \quad \mathring{H}(\text{div}; \Gamma_n, \Omega)$$

as completion under the respective graph norms of the scalar valued space $\mathring{C}^\infty(\Gamma_t, \Omega)$ and the vector valued spaces $\mathring{C}^\infty(\Gamma_t, \Omega)$, $\mathring{C}^\infty(\Gamma_n, \Omega)$, where

$$\mathring{C}^\infty(\gamma; \Omega) := \{u \in C^\infty(\overline{\Omega}) : \text{dist}(\text{supp } u, \gamma) > 0\}, \quad \gamma \in \{\Gamma, \Gamma_t, \Gamma_n\}.$$

In the latter Sobolev spaces, by Gauß' theorem the usual homogeneous scalar, tangential and normal boundary conditions

$$u|_{\Gamma_t} = 0, \quad \nu \times v|_{\Gamma_t} = 0, \quad \nu \cdot v|_{\Gamma_n} = 0$$

are generalized, where ν denotes the outward unit normal at Γ .¹³ If $\Gamma_t = \Gamma$ (and $\Gamma_n = \emptyset$) resp. $\Gamma_t = \emptyset$ (and $\Gamma_n = \Gamma$) we obtain the usual Sobolev-type spaces and write

$$\mathring{H}(\text{grad}; \Omega) = \mathring{H}^1(\Omega), \quad \mathring{H}(\text{curl}; \Omega), \quad H(\text{div}; \Omega)$$

resp.

$$H(\text{grad}; \Omega) = H^1(\Omega), \quad H(\text{curl}; \Omega), \quad \mathring{H}(\text{div}; \Omega).$$

Furthermore, we need the spaces of irrotational or solenoidal vector fields

$$\begin{aligned} H(\text{curl}_0; \Omega) &:= \{v \in H(\text{curl}; \Omega) : \text{curl } v = 0\}, \\ H(\text{div}_0; \Omega) &:= \{v \in H(\text{div}; \Omega) : \text{div } v = 0\}, \\ \mathring{H}(\text{curl}_0; \Gamma_t, \Omega) &:= \{v \in \mathring{H}(\text{curl}; \Gamma_t, \Omega) : \text{curl } v = 0\}, \\ \mathring{H}(\text{div}_0; \Gamma_n, \Omega) &:= \{v \in \mathring{H}(\text{div}; \Gamma_n, \Omega) : \text{div } v = 0\}, \end{aligned}$$

¹³Note that $\nu \times v|_{\Gamma_t} = 0$ is equivalent to $\tau \cdot v|_{\Gamma_t} = 0$ for all tangential vector fields τ at Γ_t .

where the index 0 indicates vanishing curl or div, respectively. All these spaces are Hilbert spaces. In classical terms, e.g., a vector field v belongs to $\mathring{H}(\text{curl}_0; \Gamma_t, \Omega)$ resp. $\mathring{H}(\text{div}_0; \Gamma_n, \Omega)$, if

$$\text{curl } v = 0, \quad \nu \times v|_{\Gamma_t} = 0 \quad \text{resp.} \quad \text{div } v = 0, \quad \nu \cdot v|_{\Gamma_n} = 0.$$

In [52] the crucial compact embedding

$$\mathring{H}(\text{curl}; \Gamma_t, \Omega) \cap \mathring{H}(\text{div}; \Gamma_n, \Omega) \hookrightarrow L^2(\Omega) \quad (2.1)$$

has been proved, which we refer to as Maxwell compactness property (MCP). The generalization to \mathbb{R}^N or even to Riemannian manifolds using the calculus of differential forms can be found in [63] or [50].

A first immediate consequence of (2.1) is that the space of so called ‘harmonic Dirichlet-Neumann fields’

$$\mathcal{H}(\Omega) := \mathring{H}(\text{curl}_0; \Gamma_t, \Omega) \cap \mathring{H}(\text{div}_0; \Gamma_n, \Omega) \quad (2.2)$$

is finite dimensional, since by (2.1) the unit ball is compact in $\mathcal{H}(\Omega)$. In fact, if $\Gamma_t = \emptyset$ resp. $\Gamma_t = \Gamma$, its dimension equals the first resp. second Betti number of Ω , see [96]. In classical terms we have $v \in \mathcal{H}(\Omega)$ if

$$\text{curl } v = 0, \quad \text{div } v = 0, \quad \nu \times v|_{\Gamma_t} = 0, \quad \nu \cdot v|_{\Gamma_n} = 0.$$

By an usual indirect argument we achieve another immediate and important consequence:

Lemma 1 (Maxwell Estimate for Vector Fields) *There exists a positive constant c_m , such that for all $v \in \mathring{H}(\text{curl}; \Gamma_t, \Omega) \cap \mathring{H}(\text{div}; \Gamma_n, \Omega) \cap \mathcal{H}(\Omega)^\perp$*

$$\|v\|_{L^2(\Omega)} \leq c_m \left(\|\text{curl } v\|_{L^2(\Omega)}^2 + \|\text{div } v\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

There are two options to get estimate on $\mathring{H}(\text{curl}; \Gamma_t, \Omega) \cap \mathring{H}(\text{div}; \Gamma_n, \Omega)$.

Corollary 2 (Maxwell Estimate for Vector Fields) *There exists a positive constant c_m , such that for all $v \in \mathring{H}(\text{curl}; \Gamma_t, \Omega) \cap \mathring{H}(\text{div}; \Gamma_n, \Omega)$*

$$\begin{aligned} \|(\text{id} - \pi)v\|_{L^2(\Omega)} &\leq c_m \left(\|\text{curl } v\|_{L^2(\Omega)}^2 + \|\text{div } v\|_{L^2(\Omega)}^2 \right)^{1/2}, \\ \|v\|_{L^2(\Omega)} &\leq c_m \left(\|\text{curl } v\|_{L^2(\Omega)}^2 + \|\text{div } v\|_{L^2(\Omega)}^2 + \|\pi v\|_{L^2(\Omega)}^2 \right)^{1/2}. \end{aligned}$$

Here $\pi : L^2(\Omega) \rightarrow \mathcal{H}(\Omega)$ denotes the $L^2(\Omega)$ -orthogonal projection onto Dirichlet-Neumann fields and can be expressed explicitly by

$$\pi v := \sum_{\ell=1}^L \langle v, d^\ell \rangle_{L^2(\Omega)} d^\ell, \quad \|\pi v\|_{L^2(\Omega)}^2 = \sum_{\ell=1}^L |\langle v, d^\ell \rangle_{L^2(\Omega)}|^2,$$

where $L := \dim \mathcal{H}(\Omega)$ and $(d^\ell)_{\ell=1}^L$ an $L^2(\Omega)$ -orthonormal basis of $\mathcal{H}(\Omega)$.

Here, we denote by \perp the orthogonal complement in $L^2(\Omega)$. As shown in [52] as well we have

$$\text{grad } \mathring{H}(\text{grad}; \Gamma_t, \Omega)^\perp = \mathring{H}(\text{div}_0; \Gamma_n, \Omega), \quad \text{curl } \mathring{H}(\text{curl}; \Gamma_n, \Omega)^\perp = \mathring{H}(\text{curl}_0; \Gamma_t, \Omega),$$

which implies

$$\overline{\text{grad } \mathring{H}(\text{grad}; \Gamma_t, \Omega)} = \mathring{H}(\text{div}_0; \Gamma_n, \Omega)^\perp, \quad \overline{\text{curl } \mathring{H}(\text{curl}; \Gamma_n, \Omega)} = \mathring{H}(\text{curl}_0; \Gamma_t, \Omega)^\perp,$$

where the closures are taken in $L^2(\Omega)$. Since

$$\text{grad } \mathring{H}(\text{grad}; \Gamma_t, \Omega) \subset \mathring{H}(\text{curl}_0; \Gamma_t, \Omega), \quad \text{curl } \mathring{H}(\text{curl}; \Gamma_n, \Omega) \subset \mathring{H}(\text{div}_0; \Gamma_n, \Omega)$$

we obtain by the projection theorem the Helmholtz decompositions

$$\begin{aligned} L^2(\Omega) &= \overline{\text{grad } \mathring{H}(\text{grad}; \Gamma_t, \Omega)} \oplus \mathring{H}(\text{div}_0; \Gamma_n, \Omega) \\ &= \mathring{H}(\text{curl}_0; \Gamma_t, \Omega) \oplus \overline{\text{curl } \mathring{H}(\text{curl}; \Gamma_n, \Omega)} \\ &= \overline{\text{grad } \mathring{H}(\text{grad}; \Gamma_t, \Omega)} \oplus \mathcal{H}(\Omega) \oplus \overline{\text{curl } \mathring{H}(\text{curl}; \Gamma_n, \Omega)}, \end{aligned}$$

where \oplus denotes the $L^2(\Omega)$ -orthogonal sum. Using an indirect argument, the space $\text{grad } \mathring{H}(\text{grad}; \Gamma_t, \Omega)$ is already closed by variants of Poincaré's estimate, i.e.,

$$\begin{aligned} \Gamma_t \neq \emptyset : \quad & \exists c_p > 0 \quad \forall u \in \mathring{H}(\text{grad}; \Gamma_t, \Omega) \quad \|u\|_{L^2(\Omega)} \leq c_p \|\text{grad } u\|_{L^2(\Omega)}, \\ \Gamma_t = \emptyset : \quad & \exists c_p > 0 \quad \forall u \in H(\text{grad}; \Omega) \cap \{1\}^\perp \quad \|u\|_{L^2(\Omega)} \leq c_p \|\text{grad } u\|_{L^2(\Omega)}, \end{aligned} \quad (2.3)$$

which are implied by the compact embeddings (Rellich's selection theorems)

$$\mathring{H}(\text{grad}; \Gamma_t, \Omega) \hookrightarrow L^2(\Omega), \quad H(\text{grad}; \Omega) \hookrightarrow L^2(\Omega). \quad (2.4)$$

Analogously to Corollary 2 we also have for $\Gamma_t = \emptyset$ and all $u \in H(\text{grad}; \Omega)$

$$\begin{aligned} \|u - \alpha_u\|_{L^2(\Omega)} &\leq c_p \|\text{grad } u\|_{L^2(\Omega)}, \quad \alpha_u := \lambda(\Omega)^{-1} \langle u, 1 \rangle_{L^2(\Omega)} = \oint_{\Omega} u \, d\lambda \in \mathbb{R}, \\ \|u\|_{L^2(\Omega)} &\leq c_p (\|\text{grad } u\|_{L^2(\Omega)}^2 + \|\alpha_u\|_{L^2(\Omega)}^2)^{1/2}. \end{aligned}$$

Interchanging Γ_t and Γ_n in the second equation of the latter Helmholtz decompositions and applying this Helmholtz decompositions to $\mathring{H}(\text{curl}; \Gamma_n, \Omega)$ yields the refinement

$$\text{curl } \mathring{H}(\text{curl}; \Gamma_n, \Omega) = \text{curl } \left(\mathring{H}(\text{curl}; \Gamma_n, \Omega) \cap \overline{\text{curl } \mathring{H}(\text{curl}; \Gamma_t, \Omega)} \right).$$

Now, by Lemma 1 we see that $\text{curl } \mathring{H}(\text{curl}; \Gamma_n, \Omega)$ is closed as well. We have:

Lemma 3 (Helmholtz Decompositions for Vector Fields) *The orthogonal decompositions*

$$\begin{aligned} L^2(\Omega) &= \text{grad } \mathring{H}(\text{grad}; \Gamma_t, \Omega) \oplus \mathring{H}(\text{div}_0; \Gamma_n, \Omega) \\ &= \mathring{H}(\text{curl}_0; \Gamma_t, \Omega) \oplus \text{curl } \mathring{H}(\text{curl}; \Gamma_n, \Omega) \\ &= \text{grad } \mathring{H}(\text{grad}; \Gamma_t, \Omega) \oplus \mathcal{H}(\Omega) \oplus \text{curl } \mathring{H}(\text{curl}; \Gamma_n, \Omega) \end{aligned}$$

hold. Moreover,

$$\text{curl } \mathring{H}(\text{curl}; \Gamma_n, \Omega) = \text{curl } \left(\mathring{H}(\text{curl}; \Gamma_n, \Omega) \cap \text{curl } \mathring{H}(\text{curl}; \Gamma_t, \Omega) \right).$$

2.2 Tensor fields

We extend our calculus to (3×3) -tensor (matrix) fields. For vector fields v with components in $\mathring{H}(\text{grad}; \Omega)$ and tensor fields T with rows in $\mathring{H}(\text{curl}; \Omega)$ resp. $\mathring{H}(\text{div}; \Omega)$, i.e.,

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad v_n \in \mathring{H}(\text{grad}; \Omega), \quad T = \begin{bmatrix} T_1^\top \\ T_2^\top \\ T_3^\top \end{bmatrix}, \quad T_n \in \mathring{H}(\text{curl}; \Omega) \text{ resp. } \mathring{H}(\text{div}; \Omega)$$

we define

$$\text{Grad } v := \begin{bmatrix} \text{grad}^\top v_1 \\ \text{grad}^\top v_2 \\ \text{grad}^\top v_3 \end{bmatrix} = J_v, \quad \text{Curl } T := \begin{bmatrix} \text{curl}^\top T_1 \\ \text{curl}^\top T_2 \\ \text{curl}^\top T_3 \end{bmatrix}, \quad \text{Div } T := \begin{bmatrix} \text{div } T_1 \\ \text{div } T_2 \\ \text{div } T_3 \end{bmatrix},$$

where J_v denotes the Jacobian of v and $^\top$ the transpose. We note that v and $\text{Div } T$ are vector fields, whereas T , $\text{Curl } T$ and $\text{Grad } v$ are tensor fields. The corresponding Sobolev spaces will be denoted by

$$\mathring{H}(\text{Grad}; \Omega), \quad \mathring{H}(\text{Curl}; \Omega), \quad \mathring{H}(\text{Curl}_0; \Omega), \quad \mathring{H}(\text{Div}; \Omega), \quad \mathring{H}(\text{Div}_0; \Omega)$$

and

$$\mathring{H}(\text{Grad}; \Gamma_t, \Omega), \quad \mathring{H}(\text{Curl}; \Gamma_t, \Omega), \quad \mathring{H}(\text{Curl}_0; \Gamma_t, \Omega), \quad \mathring{H}(\text{Div}; \Gamma_n, \Omega), \quad \mathring{H}(\text{Div}_0; \Gamma_n, \Omega).$$

Now, we present our three crucial tools to prove our main estimate. First we have obvious consequences from Lemmas 1 and 3:

Corollary 4 (Maxwell Estimate for Tensor Fields) *The estimate*

$$\|T\|_{L^2(\Omega)} \leq c_m \left(\|\text{Curl } T\|_{L^2(\Omega)}^2 + \|\text{Div } T\|_{L^2(\Omega)}^2 \right)^{1/2}$$

holds for all tensor fields $T \in \mathring{H}(\text{Curl}; \Gamma_t, \Omega) \cap \mathring{H}(\text{Div}; \Gamma_n, \Omega) \cap (\mathcal{H}(\Omega)^3)^\perp$. Furthermore, the analogue of Corollary 2 holds as well.

Here, $T \in \mathcal{H}(\Omega)^3$ if $T^\top = [T_1 \ T_2 \ T_3]$ with $T_m \in \mathcal{H}(\Omega)$ for $m = 1, \dots, 3$.

Corollary 5 (Helmholtz Decomposition for Tensor Fields) *The orthogonal decompositions*

$$\begin{aligned} \mathbf{L}^2(\Omega) &= \text{Grad } \mathring{\mathbf{H}}(\text{Grad}; \Gamma_t, \Omega) \oplus \mathring{\mathbf{H}}(\text{Div}_0; \Gamma_n, \Omega) \\ &= \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_t, \Omega) \oplus \text{Curl } \mathring{\mathbf{H}}(\text{Curl}; \Gamma_n, \Omega) \\ &= \text{Grad } \mathring{\mathbf{H}}(\text{Grad}; \Gamma_t, \Omega) \oplus \mathcal{H}(\Omega)^3 \oplus \text{Curl } \mathring{\mathbf{H}}(\text{Curl}; \Gamma_n, \Omega) \end{aligned}$$

hold. Moreover,

$$\text{Curl } \mathring{\mathbf{H}}(\text{Curl}; \Gamma_n, \Omega) = \text{Curl } (\mathring{\mathbf{H}}(\text{Curl}; \Gamma_n, \Omega) \cap \text{Curl } \mathring{\mathbf{H}}(\text{Curl}; \Gamma_t, \Omega)).$$

The third important tool is Korn's first inequality and a variant which meets our needs is the next lemma.

Lemma 6 (Korn's First Inequality: Standard Version) *There exists a constant $c_{\mathbf{k},s} > 0$, such that the following holds:*

(i) *If $\Gamma_t \neq \emptyset$ then*

$$(1 + c_{\mathbf{p}}^2)^{-1/2} \|v\|_{\mathbf{H}^1(\Omega)} \leq \|\text{Grad } v\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{k},s} \|\text{sym Grad } v\|_{\mathbf{L}^2(\Omega)} \quad (2.5)$$

holds for all vector fields $v \in \mathring{\mathbf{H}}(\text{Grad}; \Gamma_t, \Omega)$.

(ii) *If $\Gamma_t = \emptyset$, then the inequalities (2.5) hold for all vector fields $v \in \mathbf{H}(\text{Grad}; \Omega)$ with $\text{Grad } v \perp \mathfrak{so}(3)$ and $v \perp \mathbb{R}^3$. Moreover, the second inequality of (2.5) holds for all vector fields $v \in \mathbf{H}(\text{Grad}; \Omega)$ with $\text{Grad } v \perp \mathfrak{so}(3)$. For all $v \in \mathbf{H}(\text{Grad}; \Omega)$*

$$(1 + c_{\mathbf{p}}^2)^{-1/2} \|v - r_v\|_{\mathbf{H}^1(\Omega)} \leq \|\text{Grad } v - A_{\text{Grad } v}\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{k},s} \|\text{sym Grad } v\|_{\mathbf{L}^2(\Omega)} \quad (2.6)$$

holds, where $r_v \in \mathbf{RM}$ and $A_{\text{Grad } v} = \text{Grad } r_v$ are given by $r_v(x) := A_{\text{Grad } v}x + b_v$ and

$$A_{\text{Grad } v} := \text{skew} \oint_{\Omega} \text{Grad } v \, d\lambda \in \mathfrak{so}(3), \quad b_v := \oint_{\Omega} v \, d\lambda - A_{\text{Grad } v} \oint_{\Omega} x \, d\lambda_x \in \mathbb{R}^3.$$

We note $v - r_v \perp \mathbb{R}^3$ and $\text{Grad}(v - r_v) = \text{Grad } v - A_{\text{Grad } v} \perp \mathfrak{so}(3)$.

Proof As already mentioned in the introduction, the assertions are easy consequences of Korn's second inequality and Rellich's selection theorem for $\mathbf{H}^1(\Omega)$. \square

Remark 7 Note that $A_{\text{Grad } v} = \pi_{\mathfrak{so}(3)} \text{Grad } v$, where $\pi_{\mathfrak{so}(3)} : \mathbf{L}^2(\Omega) \rightarrow \mathfrak{so}(3)$ denotes the $\mathbf{L}^2(\Omega)$ -orthogonal projection onto $\mathfrak{so}(3)$. Thus, the assertion

$$\text{Grad}(v - r_v) = \text{Grad } v - A_{\text{Grad } v} = (\text{id} - \pi_{\mathfrak{so}(3)}) \text{Grad } v \perp \mathfrak{so}(3)$$

is trivial. Moreover, generally for $T \in \mathbf{L}^2(\Omega)$

$$\pi_{\mathfrak{so}(3)} T := A_T := \text{skew} \oint_{\Omega} T d\lambda \in \mathfrak{so}(3) \quad (2.7)$$

holds. Equivalent to (2.6) we have for all $v \in \mathbf{H}(\text{Grad}; \Omega)$

$$\begin{aligned} (1 + c_p^2)^{-1/2} \|v\|_{\mathbf{H}^1(\Omega)} &\leq (\|\nabla v\|_{\mathbf{L}^2(\Omega)}^2 + \|a_v\|_{\mathbf{L}^2(\Omega)}^2)^{1/2} \\ &\leq c_k (\|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}^2 + \|A_{\text{Grad } v}\|_{\mathbf{L}^2(\Omega)}^2 + \|a_v\|_{\mathbf{L}^2(\Omega)}^2)^{1/2} \\ &\leq c_k (\|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}^2 + \|r_v\|_{\mathbf{H}^1(\Omega)}^2)^{1/2} \end{aligned}$$

with

$$a_v = \pi_{\mathbb{R}^3} v := \oint_{\Omega} v d\lambda \in \mathbb{R}^3,$$

where $\pi_{\mathbb{R}^3} : \mathbf{L}^2(\Omega) \rightarrow \mathbb{R}^3$ denotes the $\mathbf{L}^2(\Omega)$ -orthogonal projection onto \mathbb{R}^3 . For details, we refer to the appendix.

3 Main results

We start with generalizing Korn's first inequality from gradient tensor fields to merely irrotational tensor fields.

3.1 Extending Korn's first inequality to irrotational tensor fields

Lemma 8 Let $\Gamma_t \neq \emptyset$ and $u \in \mathbf{H}(\text{grad}; \Omega)$ with $\text{grad } u \in \mathring{\mathbf{H}}(\text{curl}_0; \Gamma_t, \Omega)$. Then, u is constant on any connected component of Γ_t .

Proof It is sufficient to show that u is locally constant. Let $x \in \Gamma_t$ and $B_{2r} := B_{2r}(x)$ be an open ball of radius $2r > 0$ around x such that B_{2r} is covered by a Lipschitz-chart domain and $\Gamma \cap B_{2r} \subset \Gamma_t$. Moreover, we pick some $\varphi \in \mathring{C}^\infty(B_{2r})$ with $\varphi|_{B_r} = 1$. Then $\varphi \text{grad } u \in \mathring{\mathbf{H}}(\text{curl}; \Omega \cap B_{2r})$. Thus, the extension by zero v of $\varphi \text{grad } u$ to B_{2r} belongs to $\mathbf{H}(\text{curl}; B_{2r})$. Hence, $v|_{B_r} \in \mathbf{H}(\text{curl}_0; B_r)$. Since B_r is simply connected, there exists a $\tilde{u} \in \mathbf{H}(\text{grad}; B_r)$ with $\text{grad } \tilde{u} = v$ in B_r . In $B_r \setminus \bar{\Omega}$ we have $v = 0$. Therefore, $\tilde{u}|_{B_r \setminus \bar{\Omega}} = \tilde{c}$ with some $\tilde{c} \in \mathbb{R}$. Moreover, $\text{grad } u = v = \text{grad } \tilde{u}$ holds in $B_r \cap \Omega$, which yields $u = \tilde{u} + c$ in $B_r \cap \Omega$ with some $c \in \mathbb{R}$. Finally, $u|_{B_r \cap \Gamma_t} = \tilde{c} + c$ is constant. \square

Lemma 9 (Korn's First Inequality: Tangential Version) *Let $\Gamma_t \neq \emptyset$. There exists a constant $c_{\mathbf{k},t} \geq c_{\mathbf{k},s}$, such that*

$$\|\text{Grad } v\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{k},t} \|\text{sym Grad } v\|_{\mathbf{L}^2(\Omega)}$$

holds for all vector fields $v \in \mathbf{H}(\text{Grad}; \Omega)$ with $\text{Grad } v \in \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_t, \Omega)$.

In classical terms, $\text{Grad } v \in \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_t, \Omega)$ means that the restricted tangential traces $\nu \times \text{grad } v_n|_{\Gamma_t}$ vanish, i.e., $\text{grad } v_n = \nabla v_n$, $n = 1, \dots, 3$, are normal at Γ_t . In other words, $\tau \cdot \nabla v_n|_{\Gamma_t} = 0$ for all tangential vectors fields τ on Γ_t .

Proof Let $\tilde{\Gamma} \neq \emptyset$ be a relatively open connected component of Γ_t . Applying Lemma 8 to each component of v , there exists a constant vector $c_v \in \mathbb{R}^3$ such that $v - c_v$ belongs to $\mathring{\mathbf{H}}(\text{Grad}; \tilde{\Gamma}, \Omega)$. Then, Lemma 6 (i) (with $\Gamma_t = \tilde{\Gamma}$ and a possibly larger $c_{\mathbf{k},t}$) completes the proof. \square

Definition 10 Ω is called ‘sliceable’, if there exist $J \in \mathbb{N}$ and $\Omega_j \subset \Omega$, $j = 1, \dots, J$, such that $\Omega \setminus (\Omega_1 \cup \dots \cup \Omega_J)$ is a nullset and for $j = 1, \dots, J$

(i) Ω_j are open, disjoint and simply connected Lipschitz subdomains of Ω ,

(ii) $\Gamma_{t,j} := \text{int}_{\text{rel}}(\overline{\Omega_j} \cap \Gamma_t) \neq \emptyset$, if $\Gamma_t \neq \emptyset$.

Here, int_{rel} denotes the interior with respect to the topology on Γ .

Remark 11 Assumptions of this type are not new, see e.g. [4, p.836] or [5, p.3]. From a practical point of view, all domains considered in applications are sliceable, but it is not clear whether every Lipschitz domain is already sliceable.

Lemma 12 (Korn's First Inequality: Irrotational Version) *Let Ω be sliceable. There exists $c_{\mathbf{k}} \geq c_{\mathbf{k},t} > 0$, such that the following inequalities hold:*

(i) *If $\Gamma_t \neq \emptyset$, then for all tensor fields $T \in \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_t, \Omega)$*

$$\|T\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{k}} \|\text{sym } T\|_{\mathbf{L}^2(\Omega)}. \quad (3.1)$$

(ii) *If $\Gamma_t = \emptyset$, then for all tensor fields $T \in \mathbf{H}(\text{Curl}_0; \Omega)$ there exists a piece-wise constant skew-symmetric tensor field A such that*

$$\begin{aligned} \|T - A\|_{\mathbf{L}^2(\Omega)} &\leq c_{\mathbf{k}} \|\text{sym } T\|_{\mathbf{L}^2(\Omega)}, \\ \|T\|_{\mathbf{L}^2(\Omega)} &\leq c_{\mathbf{k}} \left(\|\text{sym } T\|_{\mathbf{L}^2(\Omega)}^2 + \|A\|_{\mathbf{L}^2(\Omega)}^2 \right)^{1/2}. \end{aligned}$$

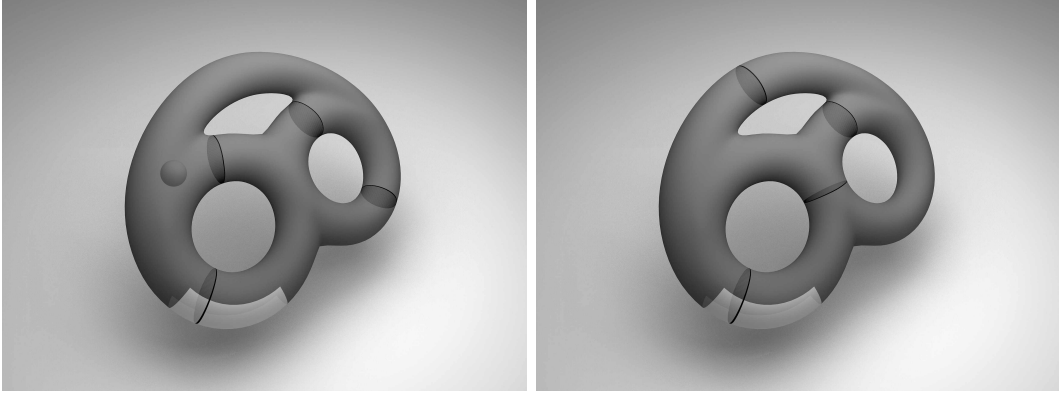


Figure 1: Two ways to cut a sliceable domain into two ($J = 2$) subdomains. Roughly speaking, a domain is sliceable if it can be ‘cut’ into finitely many simply connected Lipschitz ‘pieces’ Ω_j , i.e., any closed curve inside some piece Ω_j is homotop to a point, this is, one has to cut all handles. Holes inside Ω are permitted since we are in 3D.

(ii’) If $\Gamma_t = \emptyset$ and Ω is additionally simply connected, then (ii) holds with the uniquely determined constant skew-symmetric tensor field $A := A_T = \pi_{\mathfrak{so}(3)} T$ given by (2.7). Moreover, $T - A_T \in \mathbf{H}(\text{Curl}_0; \Omega) \cap \mathfrak{so}(3)^\perp$ and $A_T = 0$ if and only if $T \perp \mathfrak{so}(3)$. Thus, (3.1) holds for all $T \in \mathbf{H}(\text{Curl}_0; \Omega) \cap \mathfrak{so}(3)^\perp$.

Again we note that in classical terms a tensor $T \in \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_t, \Omega)$ is irrotational and the vector field $T\tau|_{\Gamma_t}$ vanishes for all tangential vector fields τ at Γ_t .

Remark 13 Without proof the last part of the result Lemma 12 (ii’) has been used implicitly in [39]. The authors of [39] neglect the problems caused by non-simply connected domains. See also our discussion in [84].

Proof Let $\Gamma_t \neq \emptyset$. According to Definition 10 we decompose Ω into $\Omega_1 \cup \dots \cup \Omega_J$. Let $T \in \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_t, \Omega)$ and $1 \leq j \leq J$. Then, the restriction $T_j := T|_{\Omega_j}$ belongs to $\mathbf{H}(\text{Curl}_0; \Omega_j)$. Picking a sequence $(T^\ell) \subset \mathring{\mathbf{C}}^\infty(\Gamma_t; \Omega)$ converging to T in $\mathbf{H}(\text{Curl}; \Omega)$, we see that $(T^\ell|_{\Omega_j}) \subset \mathring{\mathbf{C}}^\infty(\Gamma_{t,j}; \Omega)$ converges to T_j in $\mathbf{H}(\text{Curl}; \Omega_j)$. Thus, $T_j \in \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_{t,j}, \Omega_j)$. By definition, each Ω_j is simply connected. Therefore, there exist potential vector fields $v_j \in \mathbf{H}(\text{Grad}; \Omega_j)$ with $\text{Grad } v_j = T_j$. Lemma 9 yields for all j

$$\|T_j\|_{\mathbf{L}^2(\Omega_j)} \leq c_{\mathbf{k},t,j} \|\text{sym } T_j\|_{\mathbf{L}^2(\Omega_j)}$$

with $c_{\mathbf{k},t,j} > 0$. Summing up, we obtain

$$\|T\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{k}} \|\text{sym } T\|_{\mathbf{L}^2(\Omega)}, \quad c_{\mathbf{k}} := \max_{j=1,\dots,J} c_{\mathbf{k},t,j},$$

which proves (i). Now, we assume $\Gamma_t = \emptyset$. Let $T \in \mathbf{H}(\text{Curl}_0; \Omega)$ and again let Ω be decomposed into $\Omega_1 \cup \dots \cup \Omega_J$ by Definition 10. Again, since every Ω_j , $j = 1, \dots, J$, is simply connected and $T_j \in \mathbf{H}(\text{Curl}_0; \Omega_j)$, there exist vector fields $v_j \in \mathbf{H}(\text{Grad}; \Omega_j)$ with $\text{Grad } v_j =: T_j = T$ in Ω_j . By Korn's first inequality, Lemma (6) (ii), there exist positive $c_{\mathbf{k},s,j}$ and $A_{T_j} \in \mathfrak{so}(3)$ with

$$\|T_j - A_{T_j}\|_{\mathbf{L}^2(\Omega_j)} \leq c_{\mathbf{k},s,j} \|\text{sym } T_j\|_{\mathbf{L}^2(\Omega_j)}, \quad A_{T_j} = \text{skew} \oint_{\Omega_j} T_j d\lambda = \text{skew} \oint_{\Omega_j} T d\lambda.$$

We define the piece-wise constant skew-symmetric tensor field A a.e. by $A|_{\Omega_j} := A_{T_j}$ and set $c_{\mathbf{k}} := \max_{j=1,\dots,J} c_{\mathbf{k},s,j}$. Summing up gives (ii). We have also proved the first assertion of (ii'), since we do not have to slice if Ω is simply connected. The remaining assertion of (ii') concerning the projections are trivial, since $\pi_{\mathfrak{so}(3)} : \mathbf{L}^2(\Omega) \rightarrow \mathfrak{so}(3)$ is a $\mathbf{L}^2(\Omega)$ -orthogonal projector. We note that this can be seen also by direct calculations: To show that $T - A_T$ belongs to $\mathbf{H}(\text{Curl}_0; \Omega) \cap \mathfrak{so}(3)^\perp$ we note $A_T \in \mathbf{H}(\text{Curl}_0; \Omega)$ and compute for all $A \in \mathfrak{so}(3)$ (compare with (A.9))

$$\langle A_T, A \rangle_{\mathbf{L}^2(\Omega)} = \langle \int_{\Omega} T d\lambda, A \rangle_{\mathbb{R}^{3 \times 3}} = \int_{\Omega} \langle T, A \rangle_{\mathbb{R}^{3 \times 3}} d\lambda = \langle T, A \rangle_{\mathbf{L}^2(\Omega)}.$$

Hence, $A_T = 0$ implies $T \perp \mathfrak{so}(3)$. On the other hand, setting $A := A_T$ shows that $T \perp \mathfrak{so}(3)$ also implies $A_T = 0$. \square

3.2 The new inequality

From now on, we assume generally that Ω is sliceable. For tensor fields $T \in \mathbf{H}(\text{Curl}; \Omega)$ we define the semi-norm $\| \cdot \|$ by

$$\|T\|^2 := \|\text{sym } T\|_{\mathbf{L}^2(\Omega)}^2 + \|\text{Curl } T\|_{\mathbf{L}^2(\Omega)}^2. \quad (3.2)$$

Our main result is presented in the following theorem.

Theorem 14 *Let $\hat{c} := \max\{\sqrt{2}c_{\mathbf{k}}, c_{\mathbf{m}}\sqrt{1 + 2c_{\mathbf{k}}^2}\}$ and $\tilde{c} := \sqrt{2} \max\{c_{\mathbf{k}}, c_{\mathbf{m}}(1 + c_{\mathbf{k}})\} \geq \hat{c}$.*

(i) *If $\Gamma_t \neq \emptyset$, then for all tensor fields $T \in \mathring{\mathbf{H}}(\text{Curl}; \Gamma_t, \Omega)$*

$$\|T\|_{\mathbf{L}^2(\Omega)} \leq \hat{c} \|T\|. \quad (3.3)$$

(ii) *If $\Gamma_t = \emptyset$, then for all tensor fields $T \in \mathbf{H}(\text{Curl}; \Omega)$ there exists a piece-wise constant skew-symmetric tensor field A , such that*

$$\|T - A\|_{\mathbf{L}^2(\Omega)} \leq \tilde{c} \|T\|, \quad \|T\|_{\mathbf{L}^2(\Omega)} \leq \sqrt{2} \max\{\tilde{c}, 1\} (\|T\|^2 + \|A\|_{\mathbf{L}^2(\Omega)}^2)^{1/2}.$$

Note that, in general $A \notin \mathbf{H}(\text{Curl}; \Omega)$.

(ii') If $\Gamma_t = \emptyset$ and Ω is additionally simply connected, then for all tensor fields T in $\mathbf{H}(\text{Curl}; \Omega)$ there exists a uniquely determined constant skew-symmetric tensor field $A = A_T \in \mathfrak{so}(3)$, such that

$$\begin{aligned} \|T - A_T\|_{\mathbf{L}^2(\Omega)} &\leq \hat{c} \|T\|, \\ \|T\|_{\mathbf{L}^2(\Omega)} &\leq \sqrt{2} \max\{\hat{c}, 1\} (\|T\|^2 + \|A_T\|_{\mathbf{L}^2(\Omega)}^2)^{1/2}, \\ \|T - A_T\|_{\mathbf{H}(\text{Curl}; \Omega)} &\leq (1 + \hat{c}^2)^{1/2} \|T\|, \\ \|T\|_{\mathbf{H}(\text{Curl}; \Omega)} &\leq \sqrt{2} (1 + \hat{c}^2)^{1/2} (\|T\|^2 + \|A_T\|_{\mathbf{L}^2(\Omega)}^2)^{1/2}. \end{aligned}$$

and $A_T = \pi_{\mathfrak{so}(3)} T$ is given by (2.7). Moreover, $T - A_T \in \mathbf{H}(\text{Curl}; \Omega) \cap \mathfrak{so}(3)^\perp$ and $A_T = 0$ if and only if $T \perp \mathfrak{so}(3)$. Thus, (3.3) holds for all $T \in \mathbf{H}(\text{Curl}; \Omega) \cap \mathfrak{so}(3)^\perp$. Furthermore, A_T can be represented by

$$A_T = A_R := \pi_{\mathfrak{so}(3)} R = \text{skew} \oint_{\Omega} R d\lambda \in \mathfrak{so}(3),$$

where R denotes the Helmholtz projection of T onto $\mathbf{H}(\text{Curl}_0; \Omega)$ according to Corollary 5.

Proof Let $\Gamma_t \neq \emptyset$ and $T \in \mathring{\mathbf{H}}(\text{Curl}; \Gamma_t, \Omega)$. According to Corollary 5 we orthogonally decompose

$$T = R + S \in \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_t, \Omega) \oplus \text{Curl } \mathring{\mathbf{H}}(\text{Curl}; \Gamma_n, \Omega).$$

Then, $\text{Curl } S = \text{Curl } T$ and we observe that S belongs to

$$\mathring{\mathbf{H}}(\text{Curl}; \Gamma_t, \Omega) \cap \text{Curl } \mathring{\mathbf{H}}(\text{Curl}; \Gamma_n, \Omega) = \mathring{\mathbf{H}}(\text{Curl}; \Gamma_t, \Omega) \cap \mathring{\mathbf{H}}(\text{Div}_0; \Gamma_n, \Omega) \cap (\mathcal{H}(\Omega)^3)^\perp.$$

Hence, by Corollary 4 we have

$$\|S\|_{\mathbf{L}^2(\Omega)} \leq c_m \|\text{Curl } T\|_{\mathbf{L}^2(\Omega)}. \quad (3.4)$$

Then, by orthogonality, Lemma 12 (i) for R and (3.4) we obtain

$$\begin{aligned} \|T\|_{\mathbf{L}^2(\Omega)}^2 &= \|R\|_{\mathbf{L}^2(\Omega)}^2 + \|S\|_{\mathbf{L}^2(\Omega)}^2 \leq c_k^2 \|\text{sym } R\|_{\mathbf{L}^2(\Omega)}^2 + \|S\|_{\mathbf{L}^2(\Omega)}^2 \\ &\leq 2c_k^2 \|\text{sym } T\|_{\mathbf{L}^2(\Omega)}^2 + (1 + 2c_k^2) \|S\|_{\mathbf{L}^2(\Omega)}^2 \end{aligned}$$

and thus $\|T\|_{\mathbf{L}^2(\Omega)}^2 \leq \hat{c}^2 \|T\|^2$, which proves (i).

Now, let $\Gamma_t = \emptyset$ and $T \in \mathbf{H}(\text{Curl}; \Omega)$. First, we show (ii'). We follow in close lines the first part of the proof. For the convenience of the reader, we repeat the previous arguments in this special case. According to Corollary 5 we orthogonally decompose

$$T = R + S \in \mathbf{H}(\text{Curl}_0; \Omega) \oplus \text{Curl } \mathring{\mathbf{H}}(\text{Curl}; \Omega).$$

Then, $\text{Curl } S = \text{Curl } T$ and

$$S \in \mathbf{H}(\text{Curl}; \Omega) \cap \text{Curl } \mathring{\mathbf{H}}(\text{Curl}; \Omega) = \mathbf{H}(\text{Curl}; \Omega) \cap \mathring{\mathbf{H}}(\text{Div}_0; \Omega) \cap (\mathcal{H}(\Omega)^3)^\perp.$$

Again, by Corollary 4 we have (3.4). Note that $A_R \in \mathbf{H}(\text{Curl}_0; \Omega)$ since $A_R \in \mathfrak{so}(3)$ is constant. Then, by orthogonality, Lemma 12 (ii') applied to R and (3.4)

$$\begin{aligned} \|T - A_R\|_{\mathbf{L}^2(\Omega)}^2 &= \|R - A_R\|_{\mathbf{L}^2(\Omega)}^2 + \|S\|_{\mathbf{L}^2(\Omega)}^2 \leq c_{\mathbf{k}}^2 \|\text{sym } R\|_{\mathbf{L}^2(\Omega)}^2 + \|S\|_{\mathbf{L}^2(\Omega)}^2 \\ &\leq 2c_{\mathbf{k}}^2 \|\text{sym } T\|_{\mathbf{L}^2(\Omega)}^2 + (1 + 2c_{\mathbf{k}}^2) \|S\|_{\mathbf{L}^2(\Omega)}^2 \end{aligned}$$

and thus $\|T - A_R\|_{\mathbf{L}^2(\Omega)}^2 \leq \tilde{c}^2 \|T\|^2$. We need to show $A_T = A_R$ or equivalently $A_S = 0$.

For this, let $A \in \mathfrak{so}(3)$ and $S = \text{Curl } X$ with $X \in \mathring{\mathbf{H}}(\text{Curl}; \Omega)$. Then

$$\langle A_S, A \rangle_{\mathbf{L}^2(\Omega)} = \left\langle \int_{\Omega} S d\lambda, A \right\rangle_{\mathbb{R}^{3 \times 3}} = \langle \text{Curl } X, A \rangle_{\mathbf{L}^2(\Omega)} = 0.$$

By setting $A := A_S$, we get $A_S = 0$. The proof of (ii') is complete, since all other remaining assertions are trivial. Finally, we show (ii). For this, we follow the proof of (ii') up to the point, where A_R came into play. Now, by Lemma 12 (ii) for R we get a piece-wise constant skew-symmetric tensor $A := A_R$. We note that in general A does not belong to $\mathbf{H}(\text{Curl}; \Omega)$ anymore. Hence, we loose the $\mathbf{L}^2(\Omega)$ -orthogonality $R - A \perp S$. But again, by Lemma 12 (ii) and (3.4)

$$\begin{aligned} \|T - A\|_{\mathbf{L}^2(\Omega)} &\leq \|R - A\|_{\mathbf{L}^2(\Omega)} + \|S\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{k}} \|\text{sym } R\|_{\mathbf{L}^2(\Omega)} + \|S\|_{\mathbf{L}^2(\Omega)} \\ &\leq c_{\mathbf{k}} \|\text{sym } T\|_{\mathbf{L}^2(\Omega)} + (1 + c_{\mathbf{k}}) \|S\|_{\mathbf{L}^2(\Omega)} \\ &\leq c_{\mathbf{k}} \|\text{sym } T\|_{\mathbf{L}^2(\Omega)} + (1 + c_{\mathbf{k}}) c_{\mathbf{m}} \|\text{Curl } T\|_{\mathbf{L}^2(\Omega)} \end{aligned}$$

and thus $\|T - A\|_{\mathbf{L}^2(\Omega)} \leq \tilde{c} \|T\|$, which proves (ii). \square

As easy consequence we obtain:

Theorem 15 *Let $\Gamma_t \neq \emptyset$ resp. $\Gamma_t = \emptyset$ and Ω be simply connected. Then, on $\mathring{\mathbf{H}}(\text{Curl}; \Gamma_t, \Omega)$ resp. $\mathbf{H}(\text{Curl}; \Omega) \cap \mathfrak{so}(3)^\perp$ the norms $\|\cdot\|_{\mathbf{H}(\text{Curl}; \Omega)}$ and $\|\cdot\|$ are equivalent. In particular, $\|\cdot\|$ is a norm on $\mathring{\mathbf{H}}(\text{Curl}; \Gamma_t, \Omega)$ resp. $\mathbf{H}(\text{Curl}; \Omega) \cap \mathfrak{so}(3)^\perp$ and there exists a positive constant c , such that*

$$c \|T\|_{\mathbf{H}(\text{Curl}; \Omega)} \leq \|T\| = \left(\|\text{sym } T\|_{\mathbf{L}^2(\Omega)}^2 + \|\text{Curl } T\|_{\mathbf{L}^2(\Omega)}^2 \right)^{1/2} \leq \|T\|_{\mathbf{H}(\text{Curl}; \Omega)}$$

holds for all T in $\mathring{\mathbf{H}}(\text{Curl}; \Gamma_t, \Omega)$ resp. $\mathbf{H}(\text{Curl}; \Omega) \cap \mathfrak{so}(3)^\perp$.

3.3 Consequences and relations to Korn and Poincaré

There are two immediate consequences of Theorem 14 and the inclusion

$$\text{Grad } \mathring{\mathbf{H}}(\text{Grad}; \Gamma_t, \Omega) \subset \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_t, \Omega)$$

if the tensor field T is either

- irrotational or
- skew-symmetric.

For irrotational tensor fields T , i.e., $\text{Curl } T = 0$ or even $T = \text{Grad } v$, we obtain generalized versions of Korn's first inequality. E.g., in the case $\Gamma_t \neq \emptyset$ we get:

Corollary 16 (Korn's First Inequality) *Let $\Gamma_t \neq \emptyset$.*

- (i) $\|T\|_{\mathbf{L}^2(\Omega)} \leq \hat{c} \|\text{sym } T\|_{\mathbf{L}^2(\Omega)}$ holds for all tensor fields $T \in \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_t, \Omega)$.
- (ii) $\|\text{Grad } v\|_{\mathbf{L}^2(\Omega)} \leq \hat{c} \|\text{sym Grad } v\|_{\mathbf{L}^2(\Omega)}$ holds for all vector fields $v \in \mathbf{H}(\text{Grad}; \Omega)$ with $\text{Grad } v \in \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_t, \Omega)$.
- (iii) $\|\text{Grad } v\|_{\mathbf{L}^2(\Omega)} \leq \hat{c} \|\text{sym Grad } v\|_{\mathbf{L}^2(\Omega)}$ holds for all vector fields $v \in \mathring{\mathbf{H}}(\text{Grad}; \Gamma_t, \Omega)$.

These are different generalized versions of Korn's first inequality. (iii), i.e., the classical Korn's first inequality from Lemma 6 (i), is implied by (ii), i.e., Lemma 9, which is implied by (i), i.e., Lemma 12 (i). We note $c_{\mathbf{k},s} \leq c_{\mathbf{k},t} \leq c_{\mathbf{k}} \leq \hat{c}$ and that in classical terms the boundary condition, e.g., in (ii), holds, if $\text{grad } v_n = \nabla v_n$, $n = 1, \dots, 3$, are normal at Γ_t , which then extends (iii) through the weaker boundary condition.

For skew-symmetric tensors fields we get back Poincaré's inequality. More precisely, we may identify a scalar function u with a skew-symmetric tensor field T , i.e.,

$$T := T_u := \begin{bmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ -u & 0 & 0 \end{bmatrix} \cong u \quad \text{and hence} \quad \text{Curl } T = \begin{bmatrix} \partial_2 u & -\partial_1 u & 0 \\ 0 & 0 & 0 \\ 0 & -\partial_3 u & \partial_2 u \end{bmatrix}.$$

Now, $\text{Curl } T$ is as good as $\text{grad } u$, see (1.21) and

$$\nu \times T|_{\Gamma_t} = \begin{bmatrix} \nu_2 u|_{\Gamma_t} & -\nu_1 u|_{\Gamma_t} & 0 \\ 0 & 0 & 0 \\ 0 & -\nu_3 u|_{\Gamma_t} & \nu_2 u|_{\Gamma_t} \end{bmatrix} = 0 \quad \Leftrightarrow \quad u|_{\Gamma_t} = 0.$$

E.g., in the case $\Gamma_t \neq \emptyset$ we get by Theorem 14 (i):

Corollary 17 (Poincaré's Inequality) *Let $\Gamma_t \neq \emptyset$. For all special skew-symmetric tensor fields $T = T_u$ in $\mathring{\mathbf{H}}(\text{Curl}; \Gamma_t, \Omega)$, i.e., for all functions $u \in \mathring{\mathbf{H}}(\text{grad}; \Gamma_t, \Omega)$ with $u \cong T$,*

$$\|u\|_{\mathbf{L}^2(\Omega)} \leq \hat{c} \|\text{grad } u\|_{\mathbf{L}^2(\Omega)}.$$

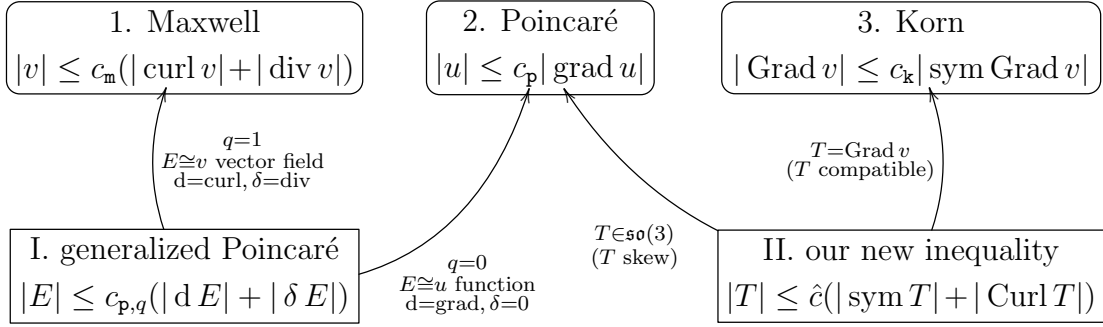


Figure 2: The three fundamental inequalities are implied by two. For the constants we have $c_p = c_{p,0}$, $c_m = c_{p,1}$ and $c_k, c_p \leq \hat{c}$.

Proof We have $T \in \mathring{H}(\text{Curl}; \Gamma_t, \Omega)$, if and only if $u \in \mathring{H}(\text{grad}; \Gamma_t, \Omega)$. Moreover,

$$2 \|u\|_{L^2(\Omega)}^2 = \|T\|_{L^2(\Omega)}^2 \leq \hat{c}^2 \|\text{Curl } T\|_{L^2(\Omega)}^2 \leq 2\hat{c}^2 \|\text{grad } u\|_{L^2(\Omega)}^2$$

holds. □

We note that the latter Corollary also remains true for general skew-symmetric tensor fields $T \in \mathring{H}(\text{Curl}; \Gamma_t, \Omega)$ and vector fields $v \in \mathring{H}(\text{Grad}; \Gamma_t, \Omega)$ with

$$T = \begin{bmatrix} 0 & -v_1 & v_2 \\ v_1 & 0 & -v_3 \\ -v_2 & v_3 & 0 \end{bmatrix} \cong v.$$

Remark 18 Let us consider the fundamental generalized Poincaré inequality for differential forms, i.e., for all $q = 0, \dots, 3$ there exist constants $c_{p,q} > 0$, such that for all q -forms $E \in \mathring{D}^q(\Gamma_t, \Omega) \cap \mathring{\Delta}^q(\Gamma_n, \Omega) \cap \mathcal{H}^q(\Omega)^\perp$

$$\|E\|_{L^{2,q}(\Omega)} \leq c_{p,q} (\|dE\|_{L^{2,q+1}(\Omega)} + \|\delta E\|_{L^{2,q-1}(\Omega)}) \quad (3.5)$$

holds. We note that the analogue of Corollary 2 holds as well. Here, E is a differential form of rank q and $d, \delta = \pm * d*$, $*$ denote the exterior derivative, co-derivative and Hodge's star operator, respectively. $D^q(\Omega)$ is the Hilbert space of all $L^{2,q}(\Omega)$ forms having weak exterior derivative in $L^{2,q+1}(\Omega)$ and by $\mathring{D}^q(\Gamma_t, \Omega)$ we denote the closure of smooth forms vanishing in a neighborhood of Γ_t with respect to the natural graph norm of $D^q(\Omega)$. The same construction is used to define the corresponding Hilbert spaces for the co-derivative $\Delta^q(\Omega)$. Moreover, we introduce $\mathcal{H}^q(\Omega) := \mathring{D}_0^q(\Gamma_t, \Omega) \cap \mathring{\Delta}_0^q(\Gamma_n, \Omega)$, the finite-dimensional space of generalized Dirichlet-Neumann forms. In classical terms, we have

$$E \in \mathcal{H}^q(\Omega) \Leftrightarrow dE = 0, \quad \delta E = 0, \quad \iota_{\Gamma_t}^* E = 0, \quad \iota_{\Gamma_n}^* * E = 0,$$

q	0	1	2	3
d	grad	curl	div	0
δ	0	div	$-\text{curl}$	grad
$\mathring{D}^q(\Gamma_t, \Omega)$	$\mathring{H}(\text{grad}; \Gamma_t, \Omega)$	$\mathring{H}(\text{curl}; \Gamma_t, \Omega)$	$\mathring{H}(\text{div}; \Gamma_t, \Omega)$	$L^2(\Omega)$
$\mathring{\Delta}^q(\Gamma_n, \Omega)$	$L^2(\Omega)$	$\mathring{H}(\text{div}; \Gamma_n, \Omega)$	$\mathring{H}(\text{curl}; \Gamma_n, \Omega)$	$\mathring{H}(\text{grad}; \Gamma_n, \Omega)$
$\iota_{\Gamma_t}^* E$	$E _{\Gamma_t}$	$\nu \times E _{\Gamma_t}$	$\nu \cdot E _{\Gamma_t}$	0
$\otimes \iota_{\Gamma_n}^* E$	0	$\nu \cdot E _{\Gamma_n}$	$-\nu \times (\nu \times E) _{\Gamma_n}$	$E _{\Gamma_n}$

Figure 3: identification table for q -forms and vector proxies in \mathbb{R}^3

where $\iota_{\Gamma_t} : \Gamma_t \hookrightarrow \Gamma \hookrightarrow \Omega$ denotes the canonical embedding.

Our new inequality, i.e., Theorem 14, together with the generalized Poincaré inequality (3.5) imply the three well known fundamental inequalities, i.e.,

1. the Maxwell inequality, i.e., Lemma 1,
2. Poincaré's inequality (2.3),
3. Korn's inequality, i.e., Lemma 6 (i).

Figure 2 illustrates this fact and Figure 3 shows an identification table for q -forms and corresponding vector proxies.

3.4 A generalization: media with structural changes

Let $\Gamma_t \neq \emptyset$ throughout this subsection. We consider the case of media with structural changes, see [54, 78]. To handle this case we use a result by Neff [75], later improved by Pompe [103], c.f., (1.16). To apply the main result from [103], let $F \in C^0(\overline{\Omega})$ be a (3×3) -matrix field satisfying $\det F \geq \mu$ with some $\mu > 0$. Then, there exists a constant $c_{k,s,F} > 0$, such that for all $v \in \mathring{H}(\text{Grad}; \Gamma_t, \Omega)$

$$\|\text{Grad } v\|_{L^2(\Omega)} \leq c_{k,s,F} \|\text{sym}(\text{Grad } v F)\|_{L^2(\Omega)}. \quad (3.6)$$

For tensor fields $T \in H(\text{Curl}; \Omega)$ we define the semi-norm $\|\cdot\|_F$ by

$$\|T\|_F^2 := \|\text{sym}(TF)\|_{L^2(\Omega)}^2 + \|\text{Curl } T\|_{L^2(\Omega)}^2. \quad (3.7)$$

Furthermore, there exists a constant $c_F > 0$ such that for all $T \in L^2(\Omega)$

$$\|TF\|_{L^2(\Omega)} \leq c_F \|T\|_{L^2(\Omega)}.$$

Let us first generalize Lemma 9.

Lemma 19 (Generalized Korn's First Inequality: Tangential Version) *There exists a constant $c_{\mathbf{k},t,F} \geq c_{\mathbf{k},s,F}$, such that the inequality*

$$\|\text{Grad } v\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{k},t,F} \|\text{sym}(\text{Grad } vF)\|_{\mathbf{L}^2(\Omega)}$$

holds for all vector fields $v \in \mathbf{H}(\text{Grad}; \Omega)$ with $\text{Grad } v \in \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_t, \Omega)$.

Proof The proof is identical with the one of Lemma 9 using (3.6) instead of Lemma 6 (i). \square

We can generalize Lemma 12.

Lemma 20 (Generalized Korn's Inequality: Irrotational Version) *There exists a constant $c_{\mathbf{k},F} \geq c_{\mathbf{k},t,F}$, such that the inequality*

$$\|T\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{k},F} \|\text{sym}(TF)\|_{\mathbf{L}^2(\Omega)}$$

holds for all tensor fields $T \in \mathring{\mathbf{H}}(\text{Curl}_0; \Gamma_t, \Omega)$.

Proof The proof is identical with the one of Lemma 12 (i) using Lemma 19 instead of Lemma 9. \square

Finally, we get:

Theorem 21 *Let $\hat{c}_F := \max\{\sqrt{2}c_{\mathbf{k},F}, c_{\mathbf{m}}\sqrt{1 + 2c_{\mathbf{k},F}^2 c_F^2}\}$. There exists $c > 0$ such that for all $T \in \mathring{\mathbf{H}}(\text{Curl}; \Gamma_t, \Omega)$*

$$\|T\|_{\mathbf{L}^2(\Omega)} \leq \hat{c}_F \|T\|_F, \quad \|T\|_{\mathbf{H}(\text{Curl}; \Omega)} \leq c \|T\|_F.$$

Proof It is sufficient to prove the first estimate. Let $T \in \mathring{\mathbf{H}}(\text{Curl}; \Gamma_t, \Omega)$. Again, we follow in close lines the proof of Theorem 14 (i). With the same notations and using Lemma 20 instead of Lemma 12 (i) we see

$$\begin{aligned} \|T\|_{\mathbf{L}^2(\Omega)}^2 &= \|R\|_{\mathbf{L}^2(\Omega)}^2 + \|S\|_{\mathbf{L}^2(\Omega)}^2 \leq c_{\mathbf{k},F}^2 \|\text{sym}(RF)\|_{\mathbf{L}^2(\Omega)}^2 + \|S\|_{\mathbf{L}^2(\Omega)}^2 \\ &\leq 2c_{\mathbf{k},F}^2 \|\text{sym}(TF)\|_{\mathbf{L}^2(\Omega)}^2 + 2c_{\mathbf{k},F}^2 \|\text{sym}(SF)\|_{\mathbf{L}^2(\Omega)}^2 + \|S\|_{\mathbf{L}^2(\Omega)}^2 \\ &\leq 2c_{\mathbf{k},F}^2 \|\text{sym}(TF)\|_{\mathbf{L}^2(\Omega)}^2 + (1 + 2c_{\mathbf{k},F}^2 c_F^2) \|S\|_{\mathbf{L}^2(\Omega)}^2 \end{aligned}$$

and thus $\|T\|_{\mathbf{L}^2(\Omega)}^2 \leq \hat{c}_F^2 \|T\|_F^2$. \square

3.5 More generalizations

Finally we note that there are a lot more generalizations. In future contributions we will also prove versions of our estimates

- in $L^p(\Omega)$ spaces (possibly just for p near to 2),
- in unbounded domains, like exterior domains,
- for domains $\Omega \subset \mathbb{R}^N$ (using differential forms),
- with inhomogeneous (restricted) tangential traces,
- concerning the deviatoric part of a tensor.

3.6 Conjectures

In view of one of the estimates which we have proved in this contribution, i.e., Theorem 14 (ii'), and the rigidity estimate (1.19) we speculate that

$$\min_{R \in \text{SO}(3)} \int_{\Omega} \text{dist}^p(T, R) \, d\lambda \leq c_{\mathbf{k}}^p \int_{\Omega} (\text{dist}^p(T, \text{SO}(3)) + |\text{Curl } T|^p) \, d\lambda$$

may hold for some $1 < p < \infty$.¹⁴

A result in Garroni et al. [39, Th.9] states that for $\Omega \subset \mathbb{R}^2$ having Lipschitz boundary¹⁵ there exists $c > 0$ such that

$$\|T\|_{L^2(\Omega)} \leq c (\|\text{sym } T\|_{L^2(\Omega)} + |\text{Curl } T|(\Omega)) \quad (3.8)$$

holds for all $T \in L^1(\Omega)$ with $A_T = 0$ and A_T from (2.7). Here, the term $|\text{Curl } T|(\Omega)$ denotes the total variation measure of the Curl-operator. However, the employed methods are restricted to the two-dimensional case since decisive use is made of the crucial \mathbb{R}^2 -identity $\text{curl}(v_1, v_2) = \text{div}(-v_2, v_1)$, see our discussion in [84].

In view of the inequality (3.8) we conjecture that for a sliceable (and maybe simply connected) domain $\Omega \subset \mathbb{R}^N$ there exists $c > 0$ such that

$$\|T\|_{L^{N/(N-1)}(\Omega)} \leq c (\|\text{sym } T\|_{L^2(\Omega)} + |\text{Curl } T|(\Omega)) \quad (3.9)$$

holds for all $T \in L^1(\Omega)$ with $A_T = 0$, where $\text{Curl } T$ is the natural generalization of the Curl-operator to higher dimensions, see [83]. This conjecture is based on the observation,

¹⁴ $\int_{\Omega} \text{dist}^p(T, \text{SO}(3)) \, d\lambda$ gives an $L^p(\Omega)$ -control of T for free, contrary to our infinitesimal version, Theorem 14 (ii').

¹⁵ The authors assume implicitly that Ω is sliceable and probably simply connected.

that for $N = 3$ and T already skew-symmetric one cannot be better than the well-known Poincaré-Wirtinger inequality in $\mathbf{BV}(\Omega)$, i.e.,

$$\|u - \alpha_u\|_{\mathbf{L}^{N/(N-1)}(\Omega)} \leq c|\nabla u|(\Omega), \quad \alpha_u := \pi_{\mathbb{R}}u = \oint_{\Omega} u \, d\lambda \in \mathbb{R}.$$

The relevance of the latter for (3.9) is clear by taking into account that for skew-symmetric matrices T , $\text{Curl } T$ can be interchanged with all partial derivatives, see inequality (1.21). However, new methods have to be developed to tackle this problem.

A Appendix

A.1 Korn's first inequality with full Dirichlet boundary condition

We note some simple estimates concerning the most elementary version of Korn's first inequality for a domain $\Omega \subset \mathbb{R}^N$, $N \in \mathbb{N}$. By twofold partial integration we get

$$\langle \partial_n v_m, \partial_m v_n \rangle_{\mathbf{L}^2(\Omega)} = \langle \partial_m v_m, \partial_n v_n \rangle_{\mathbf{L}^2(\Omega)}$$

for all smooth vector fields $v \in \mathring{\mathbf{C}}^\infty(\Omega)$ and hence

$$\begin{aligned} \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}^2 &= \frac{1}{2} \sum_{n,m=1}^N \|\partial_n v_m + \partial_m v_n\|_{\mathbf{L}^2(\Omega)}^2 \\ &= \sum_{n,m=1}^N (\|\partial_n v_m\|_{\mathbf{L}^2(\Omega)}^2 + \langle \partial_n v_m, \partial_m v_n \rangle_{\mathbf{L}^2(\Omega)}) \\ &= \|\nabla v\|_{\mathbf{L}^2(\Omega)}^2 + \|\text{div } v\|_{\mathbf{L}^2(\Omega)}^2 = \|\text{curl } v\|_{\mathbf{L}^2(\Omega)}^2 + 2\|\text{div } v\|_{\mathbf{L}^2(\Omega)}^2, \end{aligned} \tag{A.1}$$

which holds for all $v \in \mathring{\mathbf{H}}^1(\Omega)$ as well. For a quadratic matrix T and $\alpha \in \mathbb{R}$ we define the deviatoric part by

$$\text{dev}_\alpha T := T - \alpha \text{tr } T \text{ id}.$$

Then, for any $\alpha \in \mathbb{R}$ and $v \in \mathbf{H}^1(\Omega)$ we obtain for the deviatoric part of the symmetric gradient

$$\begin{aligned} \|\text{dev}_\alpha \text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}^2 &= \|\text{sym } \nabla v - \alpha \text{div } v \text{ id}\|_{\mathbf{L}^2(\Omega)}^2 \\ &= \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}^2 + c_\alpha \|\text{div } v\|_{\mathbf{L}^2(\Omega)}^2, \quad c_\alpha := \alpha(N\alpha - 2). \end{aligned} \tag{A.2}$$

Combining (A.1) and (A.2) we get for any $\alpha \in \mathbb{R}$ and $v \in \mathring{\mathbf{H}}^1(\Omega)$

$$\begin{aligned}
\|\operatorname{dev}_\alpha \operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}^2 &= \frac{1}{2} \|\nabla v\|_{\mathbf{L}^2(\Omega)}^2 + (c_\alpha + \frac{1}{2}) \|\operatorname{div} v\|_{\mathbf{L}^2(\Omega)}^2 \\
&\geq \frac{1}{2} \|\nabla v\|_{\mathbf{L}^2(\Omega)}^2 + \frac{N-2}{2N} \|\operatorname{div} v\|_{\mathbf{L}^2(\Omega)}^2, \\
\|\operatorname{dev}_\alpha \operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}^2 &= \frac{1}{2} \|\operatorname{curl} v\|_{\mathbf{L}^2(\Omega)}^2 + (c_\alpha + 1) \|\operatorname{div} v\|_{\mathbf{L}^2(\Omega)}^2 \\
&\geq \frac{1}{2} \|\operatorname{curl} v\|_{\mathbf{L}^2(\Omega)}^2 + \frac{N-1}{N} \|\operatorname{div} v\|_{\mathbf{L}^2(\Omega)}^2, \\
\|\operatorname{dev}_\alpha \operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}^2 &= (2c_\alpha + 1) \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}^2 - c_\alpha \|\nabla v\|_{\mathbf{L}^2(\Omega)}^2 \\
&\geq \frac{N-2}{N} \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}^2 - c_\alpha \|\nabla v\|_{\mathbf{L}^2(\Omega)}^2, \\
\|\operatorname{dev}_\alpha \operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}^2 &= (c_\alpha + 1) \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}^2 - \frac{c_\alpha}{2} \|\operatorname{curl} v\|_{\mathbf{L}^2(\Omega)}^2 \\
&\geq \frac{N-1}{N} \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}^2 - \frac{c_\alpha}{2} \|\operatorname{curl} v\|_{\mathbf{L}^2(\Omega)}^2.
\end{aligned}$$

We note

$$c_\alpha = \tilde{c}_\alpha - \frac{1}{N}, \quad c_\alpha + \frac{1}{2} = \tilde{c}_\alpha + \frac{N-2}{2N}, \quad c_\alpha + 1 = \tilde{c}_\alpha + \frac{N-1}{N}, \quad \tilde{c}_\alpha := N(\alpha - \frac{1}{N})^2.$$

Let us define

$$I := (0, \frac{2}{N})$$

and collect some resulting estimates: For all $\alpha \in \mathbb{R}$ and all $v \in \mathbf{H}^1(\Omega)$

$$\begin{aligned}
\|\operatorname{div} v\|_{\mathbf{L}^2(\Omega)} &\leq \sqrt{N} \|\nabla v\|_{\mathbf{L}^2(\Omega)}, \\
\|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} &\leq \|\nabla v\|_{\mathbf{L}^2(\Omega)}, \\
\|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} &\leq \|\operatorname{dev}_\alpha \operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} && \forall \alpha \in \mathbb{R} \setminus I, \\
\|\operatorname{div} v\|_{\mathbf{L}^2(\Omega)} &\leq \frac{1}{\sqrt{c_\alpha}} \|\operatorname{dev}_\alpha \operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} && \forall \alpha \in \mathbb{R} \setminus \bar{I}, \\
\|\operatorname{dev}_\alpha \operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} &\leq \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} && \forall \alpha \in \bar{I}.
\end{aligned}$$

For all $\alpha \in \mathbb{R}$ and all $v \in \mathring{\mathbf{H}}^1(\Omega)$

$$\begin{aligned}
\|\operatorname{curl} v\|_{\mathbf{L}^2(\Omega)}, \|\operatorname{div} v\|_{\mathbf{L}^2(\Omega)} &\leq \|\nabla v\|_{\mathbf{L}^2(\Omega)}, \\
\|\operatorname{div} v\|_{\mathbf{L}^2(\Omega)} &\leq \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}, \\
\|\operatorname{div} v\|_{\mathbf{L}^2(\Omega)} &\leq \sqrt{\frac{N}{N-1}} \|\operatorname{dev}_\alpha \operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}, \\
\|\nabla v\|_{\mathbf{L}^2(\Omega)} &\leq \sqrt{2} \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}, \\
\|\nabla v\|_{\mathbf{L}^2(\Omega)} &\leq \sqrt{2} \|\operatorname{dev}_\alpha \operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}, \\
\|\operatorname{dev}_\alpha \operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} &\leq \sqrt{c_\alpha + 1} \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} \quad \forall \alpha \in \mathbb{R} \setminus I, \\
\|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} &\leq \frac{1}{\sqrt{c_\alpha + 1}} \|\operatorname{dev}_\alpha \operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} \quad \forall \alpha \in \bar{I}.
\end{aligned}$$

We note that the most important case is $\alpha = 1/N \in I$, where we have $\tilde{c}_\alpha = 0$ and $c_\alpha + 1 = (N-1)/N$ as well as

$$\frac{1}{\sqrt{2}} \|\nabla v\|_{\mathbf{L}^2(\Omega)} \leq \left\| \operatorname{dev}_{\frac{1}{N}} \operatorname{sym} \nabla v \right\|_{\mathbf{L}^2(\Omega)} \leq \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} \leq \|\nabla v\|_{\mathbf{L}^2(\Omega)}$$

for all $v \in \mathring{\mathbf{H}}^1(\Omega)$.

A.2 Korn's first inequality without boundary condition

By Rellich's selection theorem for $\mathbf{H}^1(\Omega)$, Korn's second inequality and normalization one gets

$$\|\nabla v\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{k}} \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)} \quad (\text{A.3})$$

for all $v \in \mathbf{H}^1(\Omega)$ with $\nabla v \perp \mathfrak{so}(3)$. Equivalently, one has for all $v \in \mathbf{H}^1(\Omega)$

$$\begin{aligned}
\|(\operatorname{id} - \pi_{\mathfrak{so}(3)}) \nabla v\|_{\mathbf{L}^2(\Omega)} &\leq c_{\mathbf{k}} \|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}, \\
\|\nabla v\|_{\mathbf{L}^2(\Omega)} &\leq c_{\mathbf{k}} \left(\|\operatorname{sym} \nabla v\|_{\mathbf{L}^2(\Omega)}^2 + \|\pi_{\mathfrak{so}(3)} \nabla v\|_{\mathbf{L}^2(\Omega)}^2 \right)^{1/2}.
\end{aligned} \quad (\text{A.4})$$

Here $\pi_{\mathfrak{so}(3)} : \mathbf{L}^2(\Omega) \rightarrow \mathfrak{so}(3)$ denotes the $\mathbf{L}^2(\Omega)$ -orthogonal projection onto $\mathfrak{so}(3)$ and can be expressed explicitly by

$$\pi_{\mathfrak{so}(3)} T := \sum_{\ell=1}^3 \langle T, A^\ell \rangle_{\mathbf{L}^2(\Omega)} A^\ell, \quad \|\pi_{\mathfrak{so}(3)} T\|_{\mathbf{L}^2(\Omega)}^2 = \sum_{\ell=1}^3 |\langle T, A^\ell \rangle_{\mathbf{L}^2(\Omega)}|^2,$$

where $(A^\ell)_{\ell=1}^3$ is an $\mathbf{L}^2(\Omega)$ -orthonormal basis of $\mathfrak{so}(3)$. Note that $(\lambda(\Omega)^{1/2} A^\ell)_{\ell=1}^3$ is also an $\mathbb{R}^{3 \times 3}$ -orthonormal basis of $\mathfrak{so}(3)$ and thus we have the representation

$$\pi_{\mathfrak{so}(3)} T = \sum_{\ell=1}^3 \left\langle \operatorname{skew} \int_{\Omega} T \, d\lambda, A^\ell \right\rangle_{\mathbb{R}^{3 \times 3}} A^\ell = \operatorname{skew} \oint_{\Omega} T \, d\lambda =: A_T \in \mathfrak{so}(3).$$

Poincaré's inequality for vector fields by normalization reads

$$\|v\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{P}} \|\nabla v\|_{\mathbf{L}^2(\Omega)} \quad (\text{A.5})$$

for all $v \in \mathbf{H}^1(\Omega)$ with $v \perp \mathbb{R}^3$. Equivalently, one has for all $v \in \mathbf{H}^1(\Omega)$

$$\begin{aligned} \|(\text{id} - \pi_{\mathbb{R}^3})v\|_{\mathbf{L}^2(\Omega)} &\leq c_{\mathbf{P}} \|\nabla v\|_{\mathbf{L}^2(\Omega)}, \\ \|v\|_{\mathbf{L}^2(\Omega)} &\leq c_{\mathbf{P}} \left(\|\nabla v\|_{\mathbf{L}^2(\Omega)}^2 + \|\pi_{\mathbb{R}^3} v\|_{\mathbf{L}^2(\Omega)}^2 \right)^{1/2} \end{aligned} \quad (\text{A.6})$$

and hence

$$\begin{aligned} \|(\text{id} - \pi_{\mathbb{R}^3})v\|_{\mathbf{H}^1(\Omega)} &\leq (1 + c_{\mathbf{P}}^2)^{1/2} \|\nabla v\|_{\mathbf{L}^2(\Omega)}, \\ \|v\|_{\mathbf{H}^1(\Omega)} &\leq (1 + c_{\mathbf{P}}^2)^{1/2} \left(\|\nabla v\|_{\mathbf{L}^2(\Omega)}^2 + \|\pi_{\mathbb{R}^3} v\|_{\mathbf{L}^2(\Omega)}^2 \right)^{1/2}. \end{aligned}$$

Here $\pi_{\mathbb{R}^3} : \mathbf{L}^2(\Omega) \rightarrow \mathbb{R}^3$ denotes the $\mathbf{L}^2(\Omega)$ -orthogonal projection onto \mathbb{R}^3 and can be expressed explicitly by

$$\pi_{\mathbb{R}^3} v := \sum_{\ell=1}^3 \langle v, e^\ell \rangle_{\mathbf{L}^2(\Omega)} e^\ell, \quad \|\pi_{\mathbb{R}^3} v\|_{\mathbf{L}^2(\Omega)}^2 = \sum_{\ell=1}^3 |\langle v, e^\ell \rangle_{\mathbf{L}^2(\Omega)}|^2,$$

where $(e^\ell)_{\ell=1}^3$ is an $\mathbf{L}^2(\Omega)$ -orthonormal basis of \mathbb{R}^3 . Note that $(\lambda(\Omega)^{1/2} e^\ell)_{\ell=1}^3$ is also an \mathbb{R}^3 -orthonormal basis of \mathbb{R}^3 and thus we have the representation

$$\pi_{\mathbb{R}^3} v = \sum_{\ell=1}^3 \left\langle \int_{\Omega} v \, d\lambda, e^\ell \right\rangle_{\mathbb{R}^3} e^\ell = \oint_{\Omega} v \, d\lambda =: a_v \in \mathbb{R}^3.$$

Combining (A.3) and (A.5) we obtain

$$(1 + c_{\mathbf{P}}^2)^{-1/2} \|v\|_{\mathbf{H}^1(\Omega)} \leq \|\nabla v\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{K}} \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)} \quad (\text{A.7})$$

for all $v \in \mathbf{H}^1(\Omega)$ with $\nabla v \perp \mathfrak{so}(3)$ and $v \perp \mathbb{R}^3$. Without these conditions one has

$$\begin{aligned} (1 + c_{\mathbf{P}}^2)^{-1/2} \|(\text{id} - \pi_{\mathbf{RM}})v\|_{\mathbf{H}^1(\Omega)} &\leq \|(\text{id} - \pi_{\mathfrak{so}(3)})\nabla v\|_{\mathbf{L}^2(\Omega)} \leq c_{\mathbf{K}} \|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}, \\ (1 + c_{\mathbf{P}}^2)^{-1/2} \|v\|_{\mathbf{H}^1(\Omega)} &\leq \left(\|\nabla v\|_{\mathbf{L}^2(\Omega)}^2 + \|\pi_{\mathbb{R}^3} v\|_{\mathbf{L}^2(\Omega)}^2 \right)^{1/2} \\ &\leq c_{\mathbf{K}} \left(\|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}^2 + \|\pi_{\mathfrak{so}(3)} \nabla v\|_{\mathbf{L}^2(\Omega)}^2 + \|\pi_{\mathbb{R}^3} v\|_{\mathbf{L}^2(\Omega)}^2 \right)^{1/2} \\ &\leq c_{\mathbf{K}} \left(\|\text{sym } \nabla v\|_{\mathbf{L}^2(\Omega)}^2 + \|\pi_{\mathbf{RM}} v\|_{\mathbf{H}^1(\Omega)}^2 \right)^{1/2}, \end{aligned} \quad (\text{A.8})$$

for all $v \in \mathbf{H}^1(\Omega)$, where $\pi_{\mathbf{RM}} : \mathbf{H}^1(\Omega) \rightarrow \mathbf{RM}$ is defined by

$$\pi_{\mathbf{RM}} v := (\pi_{\mathfrak{so}(3)} \nabla v) \xi + \pi_{\mathbb{R}^3} (v - (\pi_{\mathfrak{so}(3)} \nabla v) \xi) = \pi_{\mathbb{R}^3} v + (\text{id} - \pi_{\mathbb{R}^3})((\pi_{\mathfrak{so}(3)} \nabla v) \xi)$$

with the identity function $\xi(x) := \text{id}(x) = x$. We note

$$\begin{aligned} \pi_{\mathfrak{so}(3)} \nabla v &= A_{\nabla v} = \text{skew} \oint_{\Omega} \nabla v \, d\lambda \in \mathfrak{so}(3), & \pi_{\mathbb{R}^3} v &= a_v = \oint_{\Omega} v \, d\lambda \in \mathbb{R}^3, \\ \pi_{\text{RM}} v &= r_v := A_{\nabla v} \xi + a_v - A_{\nabla v} a_{\xi} \in \text{RM}, & \nabla \pi_{\text{RM}} v &= \nabla r_v = A_{\nabla v} = \pi_{\mathfrak{so}(3)} \nabla v \in \mathfrak{so}(3). \end{aligned}$$

Note that $u := (\text{id} - \pi_{\text{RM}})v = v - r_v$ belongs to $H^1(\Omega)$ and satisfies

$$\nabla u = (\text{id} - \pi_{\mathfrak{so}(3)}) \nabla v \perp \mathfrak{so}(3), \quad u = (\text{id} - \pi_{\mathbb{R}^3})(v - (\pi_{\mathfrak{so}(3)} \nabla v) \xi) \perp \mathbb{R}^3.$$

Hence (A.7) holds for u . Moreover, we have for $v \in H^1(\Omega)$

$$\pi_{\text{RM}} v = 0 \quad \Leftrightarrow \quad \pi_{\mathfrak{so}(3)} \nabla v = 0 \wedge \pi_{\mathbb{R}^3} v = 0 \quad \Leftrightarrow \quad \nabla v \perp \mathfrak{so}(3) \wedge v \perp \mathbb{R}^3.$$

This can also be seen by elementary calculations: For all $A \in \mathfrak{so}(3)$ and all $a \in \mathbb{R}^3$ we have

$$\begin{aligned} \langle A_{\nabla v}, A \rangle_{L^2(\Omega)} &= \left\langle \int_{\Omega} \nabla v \, d\lambda, A \right\rangle_{\mathbb{R}^{3 \times 3}} = \int_{\Omega} \langle \nabla v, A \rangle_{\mathbb{R}^{3 \times 3}} \, d\lambda = \langle \nabla v, A \rangle_{L^2(\Omega)}, \\ \langle r_v, a \rangle_{L^2(\Omega)} &= \left\langle \int_{\Omega} r_v \, d\lambda, a \right\rangle_{\mathbb{R}^3} = \left\langle A_{\nabla v} \int_{\Omega} \xi \, d\lambda + \lambda(\Omega)(a_v - A_{\nabla v} a_{\xi}), a \right\rangle_{\mathbb{R}^3} \\ &= \left\langle \int_{\Omega} v \, d\lambda, a \right\rangle_{\mathbb{R}^3} = \langle v, a \rangle_{L^2(\Omega)}. \end{aligned} \tag{A.9}$$

Thus $\nabla u \perp \mathfrak{so}(3)$ and $u \perp \mathbb{R}^3$. This shows also that $\nabla v \perp \mathfrak{so}(3)$ and $v \perp \mathbb{R}^3$ if we have $r_v = 0$. On the other hand, if $v \in H^1(\Omega)$ with $\nabla v \perp \mathfrak{so}(3)$ and $v \perp \mathbb{R}^3$, then $r_v = 0$ because $A_{\nabla v} = 0$ by setting $A := A_{\nabla v} \in \mathfrak{so}(3)$ and then $a_v = 0$ by setting $a := a_v = r_v$.

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